15.7 INFINITE SERIES OF FUNCTIONS (Kanpur

We now consider series whose terms are functions defined in some set S. Let

$$f_1 + f_2 + \dots + f_n + \dots$$

be such a series. We write

$$S_n(x) = f_1 + f_2 + \dots + f_n$$

so that $\{S_n(x)\}$ is a sequence of functions.

We say that the series is point-wise convergent if the sequence $\{S_n(x)\}$ is point-wise convergent. Also the series is said to be uniformly convergent, if the sequence $\{S_n(x)\}$ is uniformly convergent. Also, the point-wise limit or the uniform limit of $\{S_n(x)\}$ as the case may be is said to be the point-wise sum or the uniform sum of the series and is denoted by $S(x)$.

15.8 TEST FOR THE UNIFORM CONVERGENCE OF A SERIES

15.8.1 Cauchy's General Principle of Convergence

The necessary and sufficient condition for the uniform convergence in $[a, b]$ of a series Σf_n is that to every positive number, ε, there corresponds a positive integer m such that $\forall\, n \geq m$ and $\forall\, x \in [a,b]$,

$$\left| f_{n+1}(x) + f_{n+2}(x) + \dots + f_{n+p}(x) \right| < \varepsilon.$$

This result is an immediate consequence of the corresponding result for sequences proved in Art. 15.2.

15.8.2. Weierstrass's M-test for Uniform Convergence

(Meerut 2006, 08, 09; Kanpur 2007, 09; Agra 2007; Delhi Maths (Prog) 20

Theorem. A series Σf_n will converge uniformly in $[a, b]$, if there exists a convergent series ΣM_n of positive numbers such that $\forall\ x \in [a,b]$,

Scanned with CamScanner

$$|f_n(x)| \le M_n$$

Proof. Let ε be a positive number. Since ΣM_n is convergent, there exists a positive integer m such that

$$|M_{n+1} + M_{n+1} + \ldots + M_{n+p}| < \varepsilon, \ \forall n \ge m \text{ and } \forall p \ge 0. \quad \ldots (i)$$

Also, given that $\quad |f_n(x)| \le M_n \ \forall x \in [a,b] \quad \ldots (ii)$

From (*i*) and (*ii*), we see that

$$\forall x \in [a, b], \forall n \ge m \text{ and } p \ge 0,$$

$$|f_{n+1}(x) + f_{n+2}(x) + \ldots + f_{n+p}(x)| \le [M_{n+1} + M_{n+2} + \ldots + M_{n+p}] < \varepsilon.$$

Hence Σf_n is uniformly convergent in $[a, b]$

EXAMPLES

Ex. 1. *Show that the following series are uniformly convergent for real values of x :*

(*i*) $\sum_{n=1}^{\infty} \frac{\sin nx}{n^p}, p > 1$ (*I.A.S. 1998*) ; $\quad \sum_{n=1}^{\infty} \frac{\cos nx}{n^4}, x \in R$ (*Delhi B.Sc. (Prog) 2008*)

(*ii*) $\sum_{n=1}^{\infty} \frac{\cos nx}{n^2}$ (*Kanpur 2005; Meerut 2009*) $\quad$ (*iii*) $\sum_{n=1}^{\infty} \frac{\sin nx}{n^2}$ (*Delhi Maths (H) 2001*)

(*iv*) $\sum_{n=1}^{\infty} \frac{\sin (x^2 + nx^2)}{n(n+1)}$ (*Delhi Maths (H) 2001*)

Sol. (*i*) Here, $f_n(x) = (\sin nx) / n^p$ and so

$$|f_n(x)| = \left|\frac{\sin nx}{n^p}\right| = \frac{|\sin nx|}{n^p} \le \frac{1}{n^p} = M_n, \text{say } \forall x \in R$$

Since $\sum_{n=1}^{\infty} M_n = \sum_{n=1}^{\infty} \frac{1}{n^p}$ is convergent for p > 1, so by Weierstrass's *M* test, the given series converges uniformly for all real values of *x*.

(*ii*) and (iii). Left as exercises for the reader.

(*iv*) Here $f_n(x) = \{\sin (x^2+nx^2)\}/n(n+1)$

$$\therefore |f_n(x)| = \left|\frac{\sin(x^2 + nx^2)}{n(n+1)}\right| = \frac{|\sin(x^2 + nx^2)|}{n^2(1+1/n^2)} \le \frac{1}{n^2} = M_n \text{ for } \forall x \in R$$

Since $\sum_{n=1}^{\infty} M_n = \sum_{n=1}^{\infty} \frac{1}{n^2}$ is convergent, so by Weierstrass's M-test, the given series is uniformly convergent for all real values of *x*.

Ex. 2. *Show that the series* $\sum_{n=1}^{\infty} \frac{x}{n(1+nx^2)}$ *converges.* (*Kanpur 2007*)

(*Agra 1998, 99, 2001*); *Delhi Maths (H) 2000; Delhi B.Sc. Physics (H) 1997, 98*) ..(1)

Sol. Here $\quad f_n(x) = x/n(1+nx^2) \quad$ ...(2)

$$\therefore \quad \frac{df_n(x)}{dx} = \frac{1}{n} \cdot \frac{(1+nx^2)\cdot 1 - x \cdot 2nx}{(1+nx^2)^2} = \frac{1-nx^2}{n(1+nx^2)^2}$$

For max, or min, $\quad d f_n(x)/dx = 0 \Rightarrow 1 - nx^2 = 0 \Rightarrow x = 1/\sqrt{n}$

From (2), $$\frac{d^2 f_n(x)}{dx^2} = \frac{1}{n} \times \frac{(1+nx^2)^2 \cdot (-2nx) - (1-nx^2) \cdot 2(1+nx^2) \cdot 2nx}{(1+nx^2)^4}$$

$$\therefore \quad \frac{d^2 f_n(x)}{dx^2} = -\frac{2x\{(1+nx^2)+2(1-nx^2)\}}{(1+nx^2)^3} \quad \text{and} \quad \left[\frac{d^2 f_n(x)}{dx^2}\right]_{x=1/\sqrt{n}} = -\frac{1}{2\sqrt{n}} < 0,$$

showing that $f_n(x)$ is maximum at $x = 1\sqrt{n}$ and the maximum value of $f_n(x)$ from (1) is

$$\frac{1/\sqrt{n}}{n(1+1)}, \quad i.e., \quad \frac{1}{2n^{3/2}}$$

Henc $$| f_n(x) | \leq \frac{1}{2n^{3/2}} < \frac{1}{n^{3/2}} = M_n, \text{ say}$$

Since $\sum_{n=1}^{\infty} M_n = \sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$ is conversent, so by Weierstrass's M-test, the given series is uniform convergent for all real values of x.