DEPARTMENT OF PHYSICS AND ASTROPHYSICS
UNIVERSITY OF DELHI

B.Sc.(H) PHYSICS (CBCS)
SEMESTER - VI

STATISTICAL MECHANICS

Model Problem Set

1 Classical Statistics

1.1 Macrostate and Microstate, Phase Space, Ensemble, Thermodynamic Probability

Problem 1: Consider a particle undergoing simple harmonic motion such that the position of the particle changes with time as $x = x_0 \cos(\omega t + \phi)$, where the phase ϕ is completely unknown, and therefore the position of the oscillator is not known. One therefore has to resort to determining the probability that the position of the oscillator lies between x and $x + dx$.

(a) This probability must be proportional to the time the oscillator spends between x and $x + dx$. Find the speed of the oscillator at position x as a function of x, ω and x_0. Using this expression, determine the probability $p(x)dx$ that the position of the oscillator is between x and $x + dx$.

(b) Let the energy of the oscillator lie between E and $E + \Delta E$, where $\Delta E << E$. Sketch the phase space and the region accessible to the particle, calculating the volume of the accessible region. Next, compute the ratio of the volume of the accessible phase space corresponding to the position of the particle lying between x and $x + dx$ and the total volume of the accessible phase space. What does this result signify?

Solution 1:

(a) *The energy of the oscillator is $E = m\omega^2 x_0^2/2$. If the position of the oscillator is x and its speed is v, then*

$$\begin{aligned} \frac{1}{2}mv^2 + \frac{1}{2}m\omega^2 x^2 &= \frac{1}{2}m\omega^2 x_0^2 \\ \implies v &= \omega\sqrt{x_0^2 - x^2} \end{aligned}$$

Since $dx/dt = v$, therefore the time it spends between x and $x + dx$ is

$$\begin{aligned} dt &= \frac{dx}{v} \\ &= \frac{dx}{\omega\sqrt{x_0^2 - x^2}} \end{aligned}$$

The probability that the position is between x and $x + dx$ is then of the form

$$P(x)dx = \frac{N dx}{\omega\sqrt{x_0^2 - x^2}}$$

where N is a normalisation constant, to be determined by the constraint $\int_{-x_0}^{x_0} dx P(x) = 1$. Evaluating the integral gives $N = \omega/\pi$. Then

$$P(x)dx = \frac{dx}{\pi\sqrt{x_0^2 - x^2}}$$

(b) *Given the relation*

$$\frac{p^2}{2m} + \frac{1}{2}m\omega^2 x^2 = E$$

the phase space trajectory corresponding to energy E forms an ellipse in the $p - x$ plane, with the region of the phase space corresponding to energy lying between E and $E + \Delta E$ being the region between two ellipses. To compute the volume of this region, we first compute the area enclosed by the *elliptical region corresponding to the energy lying between zero and E. Given the above equation, this volume (area, in this case) is easily computed to be*

$$\Gamma = \frac{2\pi E}{\omega}$$

Then the volume of the region corresponding to energy lying between E and $E + \Delta E$ is

$$\Delta\Gamma = \frac{2\pi \Delta E}{\omega}$$

The volume of the region corresponding to position lying between x and $x + dx$ is

$$\delta\Gamma = dxdp$$

where dp is computed using the relation between p, x and E. For a given x, p has two values, one positive and one negative. The volume is then twice the volume computed assuming that p is positive, given by

$$p = \sqrt{2mE - m^2\omega^2 x^2}$$

Then

$$dp = \frac{m\Delta E}{\sqrt{2mE - m^2\omega^2 x^2}}$$

Using the fact that $E = m\omega^2 x_0^2/2$, we get

$$\delta\Gamma = \frac{2\Delta E}{\omega}\frac{dx}{\sqrt{x_0^2 - x^2}}$$

Then

$$\frac{\delta\Gamma}{\Delta\Gamma} = \frac{dx}{\pi\sqrt{x_0^2 - x^2}}$$

which is the same as the probability of the oscillator's position lying between x and $x + dx$.

Problem 2: Consider an isolated system of four non-interacting spins labelled 1, 2, 3, and 4, each with magnetic moment m, interacting with an external magnetic field B. Each spin can be parallel ('up') or antiparallel ('down') to B, with the energy of a spin parallel to B equal to $\epsilon = -mB$ and the energy of a spin antiparallel to B equal to $\epsilon = +mB$. Let the total energy of the system be $E = -2mB$.

(a) How many microstates of the system correspond to this macrostate? Enumerate these microstates.

(b) What is the probability that the system is in a given microstate in equilibrium?

(c) What is the probability that a given spin points up? Use this probability to campute the mean magnetic moment of a given spin in equilibrium.

(d) What is the probability that if spin 1 is 'up', spin 2 is also 'up'?

Solution 2: *Since the energy of the system is $E = -2mB$, the total magnetic moment of the system is $M = +2m$, which corresponds to one spin 'down' and three spins 'up'.*

(a) *The total number of microstates is then*

$$\begin{aligned} \Omega &= \frac{4!}{3! \times 1!} \\ &= 4 \end{aligned}$$

(b) *The probability of any given microstate is*

$$P_r = \frac{1}{\Omega} = \frac{1}{4}$$

(c) *Given that a particular spin is 'up', there are three microstates corresponding to this (corresponding to one of the remaining spins being 'down'). Therefore, the probability is*

$$P_u = \frac{3}{4}$$

and the probability of the given spin being 'down' is

$$P_d = \frac{1}{4}$$

The mean magnetic moment of a give spin is

$$\overline{m} = m \times P_u + (-m) \times P_d = \frac{m}{2}$$

(d) *Given that spin 1 is up, there are in all three microstates corresponding to this. Of these, two correspond to spin 2 also being up. Therefore, the propbability that spin 2 is also up is*

$$P_{1u|2u} = \frac{2}{3}$$

Problem 3: Consider a system of four non-interacting distinguishable particles, with each particle localised to a lattice site. The energy of each particle is is restricted to values $\epsilon = 0, \epsilon_0, 2\epsilon_0, 3\epsilon_0, ...$ The system is divided into two subsystems A and B, subsystem A consisting of particles 1 and 2, and B consisting of particles 3 and 4 respectively. A and B are initially thermally insulated from each other, with energies $E_A = 5\epsilon_0$ and $E_B = \epsilon_0$. What are the possible microstates of the composite system? Now, suppose the two subsystems are allowed to thermally interact with each other, so that they can exchange energy without the total energy of the system changing. After equilibrium is attained, enumerate the possible microstates of the composite system. In equilibrium, what is the probability that subsystem A has energy E_A, for $E_A = 0, \epsilon_0, 2\epsilon_0, .., 6\epsilon_0$? For what value of E_A is the probability maximum?

Solution 3: *Let us denote the microstates of the composite system as* $(-, -|-, -)$ *where the first two slots are for system A and the second two slots for system B. Given that system A has energy* $5\epsilon_0$ *and system B has energy* ϵ_0 *gives the following 12 microstates for the composite system:*
$(0, 5|0, 1), (0, 5|1, 0), (5, 0|0, 1), (5, 0|1, 0), (1, 4|0, 1), (1, 4|1, 0), (4, 1|0, 1), (4, 1|1, 0), (2, 3|0, 1), (2, 3|1, 0), (3, 2|0, 1)$ *and* $(3, 2|1, 0)$, *where all numbers are in units of* ϵ_0. *Once the system reaches equilibrium, the total number of microstates is given by the total number of ways of distributing energy* $6\epsilon_0$ *among* 4 *particles.*

$$\Omega = \frac{(6+4-1)!}{6!(4-1)!} = \frac{9!}{6!3!} = 84$$

The probability that system A has energy E_A *is given by the product of the number of microstates of system A corresponding to its energy being* E_A *and the number of microstates of system B corresponding* *to its energy being $E_B = E - E_A$, divided by the total number of microstates (here, $E = 6\epsilon_0$). For example, let $E_A = \epsilon$. Then, the number of microstates accessible to A corresponding to this is 2. The energy available to system B is 5ϵ, which can be distributed among 2 particles in 6 different ways. Then, the total number of microstates corresponding to system A having this energy is $\Omega(E_A = \epsilon_0) = 2 \times 6 = 12$. The probability of this is then*

$$\begin{aligned} P(E_A = \epsilon_0) &= \frac{\Omega(E_A = \epsilon_0)}{\Omega} \\ &= \frac{12}{84} \\ &= \frac{1}{7} \end{aligned}$$

Similarly, probabilities of other possible values of energy can be computed, and it can be checked that the probability is maximum for $E_A = 3\epsilon_0$, and is equal to $4/21$.

1.2 Entropy and Thermodynamic Probability

Problem 1: Consider a system of N particles (which could be interacting with each other) with energy E and occupying a volume V. The entropy of the system is known to be extensive. Suppose the energy of the system is changed, such that the new energy is λE, where λ is a multiplicative factor. Can you say that the new entropy will be λS, where S is the original entropy? If not, what other changes will be needed such that this is true?

Solution 1: *Since entropy is extensive, therefore, entropy will be λS under $N \to \lambda N, V \to \lambda V$ and $E \to \lambda E$ simultaneously.*

Problem 2: Consider a system of $N >> 1$ weakly interacting particles, each of which can be in quantum states with energies $0, \epsilon, 2\epsilon, 3\epsilon, ...$ Given the system has a certain energy, the temperature of the system is given by

$$\begin{aligned} \frac{1}{T} &= \frac{\partial S}{\partial E} \\ &\simeq \frac{\Delta S}{\Delta E} \end{aligned}$$

where ΔS is the change in the entropy of the system due to the change in the energy of the system by ΔE.

(a) If the system is in its ground state, what is its entropy?

(b) If the total energy of the system is ϵ, what is its entropy?

(c) What is the change in entropy of the system if the total energy of the system is increased from ϵ to 2ϵ?

(d) Given the above definition of temperature, what is the temperature of the system if its total energy is ϵ?

Solution 2:

(a) *The ground state corresponds to a unique microstate, in which all the particles have energy 0. Its entropy, therefore, is zero.*

(b) *If the total energy is ϵ, there are N possible microstates corresponding to this, in which one particle has energy ϵ and others have energy 0. The entropy is $S = k_B \ln N$.*

(c) *If the total energy is 2ϵ, this corresponds to either two particles with energy ϵ (which corresponds to $N(N-1)/2$ possible microstates) or one particle with energy 2ϵ (which corresponds to N possible microstates). Therefore, the total number of microstates is $\Omega = N + N(N-1)/2 = N(N+1)/2$. Then, the entropy of the system is*

$$\begin{aligned} S' &= k_B \ln\left(N(N+1)/2\right) \\ &\simeq k_B \ln N + k_B \ln(N/2) \quad (\because N >> 1) \end{aligned}$$

Then, the change in entropy is $\Delta S = k_B \ln(N/2)$.

(d) *The temperature of the system is given by*

$$\begin{aligned} \frac{1}{T} &\simeq \frac{\Delta S}{\Delta E} \\ &= \frac{k_B}{\epsilon} \ln(N/2) \end{aligned}$$

Therefore,

$$T = \frac{\epsilon}{k_B} \frac{1}{\ln(N/2)}$$

Problem 3: A system of four weakly interacting distinct particles is such that each particle can be in one of four states with energies $\epsilon, 2\epsilon, 3\epsilon$ and 4ϵ respectively. If the system has total energy 15ϵ, what is the entropy of the system? For what possible values of total energy is the entropy of the system zero?

Solution 3: Since the total energy of the system is 15ϵ, this corresponds to three particles with energy 4ϵ and one particle with energy 3ϵ. There are four microstates corresponding to this. Therefore, the entropy of the system is $S = k_B \ln 4$. The entropy of the system is zero if either the total energy of the system is 4ϵ or 16ϵ, since these macrostates correspond to one microstate each.

Problem 4: Consider a lattice of N non-interacting distinguishable particles, with each particle localised to a lattice site. The energy of each particle is restricted to values $\epsilon = 0, \epsilon_0, 2\epsilon_0, 3\epsilon_0, ...$ The system is in equilibrium.

(a) If the energy of the system is E, what is the number of microstates of the system?

(b) Find an expression for the entropy of the system as a function of energy and simplify it using Sterling's approximation $\ln n \simeq n \ln n - n$ for $n >> 1$.

(c) Using the relation

$$\frac{1}{T} = \frac{\partial S}{\partial E}$$

determine a relation between the energy of the system and its temperature.

Hint: The problem of determining the number of microstates can be reduced to counting the number of ways of arranging a certain number of sticks and a certain number of dots along a line.

Solution 4:

(a) *A microstate of the system can be represented as the set* $(n_1, n_2, .., n_N)$ *which represents a state in which the first particle has energy* $n_1\epsilon_0$*, second has energy* $n_1\epsilon_0$*,... Let* $M = E/\epsilon$*. Then,*

$$n_1 + n_2 + ..n_N = M$$

The total number of microstates is the number of ways of choosing integers $n_1, n_2, .., n_N$ *such that their sum is* M*. This is the same as the number of ways of partitioning* M *into* N *parts. This can be visualised as the number of ways* M *dots and* $N-1$ *sticks can be arranged in a line, the sticks creating partitions. Therefore, the number of microstates is*

$$\begin{aligned}\Omega(E) &= \frac{(M+N-1)!}{M!(N-1)!} \\ &= \frac{(E/\epsilon+N-1)!}{(E/\epsilon)!(N-1)!} \\ &\simeq \frac{(E/\epsilon+N)!}{(E/\epsilon)!N!}\end{aligned}$$

assuming $E, N >> 1$*.*

(b) *The entropy of the system is*

$$\begin{aligned}S &= k_B \ln\Omega \\ &\simeq k_B\left[\left(\frac{E}{\epsilon}+N\right)\ln\left(\frac{E}{\epsilon}+N\right) - \frac{E}{\epsilon}\ln\frac{E}{\epsilon} - N\ln N\right]\end{aligned}$$

where Sterling's approximation has been used.

(c) *The temperature of the system is given by*

$$\begin{aligned}\frac{1}{T} &= \frac{\partial S}{\partial E} \\ &= \frac{k_B}{\epsilon}\ln\left(1+\frac{N\epsilon}{E}\right)\end{aligned}$$

which is inverted to give

$$E = \frac{N\epsilon}{e^{\epsilon/k_BT}-1}$$

1.3 Maxwell Boltzmann Distribution

Problem 1: Consider atomic hydrogen in thermal equilibrium at temperature T. Estimate the ratio of the number of atoms with energy $E = -3.4$ eV to the number of atoms with energy $E = -13.6$ eV for $T = 1000°K$.

Solution 1: It is useful to determine the temperature equivalent to energy equal to 1ev. This temperature is

$$\begin{aligned}T_{ev} &= \frac{1 \text{ ev}}{k_B} \\ &= 1.16\times 10^4 \,{}^\circ K\end{aligned}$$

There are two microstates corresponding to the atom in its ground state (with energy -13.6 eV) corresponding to the two spin states of the electron. Similarly, there are eight microstates corresponding to the first excited states (with energy −3.4 eV). Then, the relative probability of the atom being in one of the first excited states is

$$\begin{aligned}\frac{P_{exc}}{P_{gr}} &= \frac{8\times e^{3.4ev/k_BT}}{2\times e^{13.6ev/k_BT}}\\ &= 4e^{-10.2ev/k_BT}\\ &= 4e^{-10.2\ T_{ev}/T}\\ &= 4e^{-118.32}\end{aligned}$$

which is vanishingly small.

Problem 2: A system of N weakly interacting particles, each of mass m, is in thermal equilibrium at temperature T. The system is contained in a cubical box of side L, whose top and bottom surfaces are parallel to the Earth's surface, where the acceleration due to gravity is g. A coordinate system is set up with the origin at the centre of the base of the box and the positive z axis along the vertical direction, such that the ranges of coordinates accessible to any particle are $-L/2 \le x \le L/2$, $-L/2 \le y \le L/2$, $0 \le z \le L$.

(a) What is the probability that a given particle has velocity in the range (v_x, v_y, v_z) and $(v_x + dv_x, v_y + dv_y, v_z + dv_z)$?

(b) What is the probability that a given particle has x coordinate between x and $x + dx$?

(c) What is the probability that a given particle has y coordinate between y and $y + dy$?

(d) What is the probability that a given particle has z coordinate between z and $z + dz$?

(e) From the above probability distributions, calculate the mean kinetic and potential energies of a particle.

Solution 2: *The probability of a microstate of a particle corresponding to its position lying between* (x, y, z) *and* $(x + dx, y + dy, z + dz)$ *and velocity between* (v_x, v_y, v_z) *and* $(v_x + dv_x, v_y + dv_y, v_z + dv_z)$ *is given by the Maxwell Boltzmann distribution*

$$P(x,y,z,v_x,v_y,v_z)dxdydzdv_xdv_ydv_z = Ne^{-\beta[m(v_x^2+v_y^2+v_z^2)/2+mgz]}$$

where N *is a normalisation constant such that* $\int Pdxdydzdv_xdv_ydv_z = 1$. *The integral over velocity components is Gaussian and gives* $(2\pi/m\beta)^{3/2}$. *The integral over position coordinates gives*

$$\begin{aligned}I &= \int_{-L/2}^{L/2} dxdy \int_0^L dze^{-\beta mgz}\\ &= \frac{L^2}{\beta mg}\left(1-e^{-\beta mgL}\right)\end{aligned}$$

Then, the normalisation constant is determined to be

$$N = \left(\frac{m\beta}{2\pi}\right)^{3/2} \frac{\beta mg}{L^2\left(1-e^{-\beta mgL}\right)}$$

(a)

$$\begin{aligned}P(v_x,v_y,v_z)dv_xdv_ydv_z &= dv_xdv_ydv_z\int dxdydzP(x,y,z,v_x,v_y,v_z)\\ &= \left(\frac{m\beta}{2\pi}\right)^{3/2} e^{-\beta m(v_x^2+v_y^2+v_z^2)/2}dv_xdv_ydv_z\end{aligned}$$

which is just the Maxwell velocity distribution.

(b)

$$\begin{aligned} P(x)dx &= dx \int dydzdv_xdv_ydv_z P(x,y,z,v_x,v_y,v_z) \\ &= \frac{dx}{L} \end{aligned}$$

(c)

$$\begin{aligned} P(y)dy &= dy \int dxdzdv_xdv_ydv_z P(x,y,z,v_x,v_y,v_z) \\ &= \frac{dy}{L} \end{aligned}$$

(d)

$$\begin{aligned} P(z)dz &= dz \int dxdydv_xdv_ydv_z P(x,y,z,v_x,v_y,v_z) \\ &= \frac{\beta mg}{(1-e^{-\beta mgL})} e^{-\beta mgz} dz \end{aligned}$$

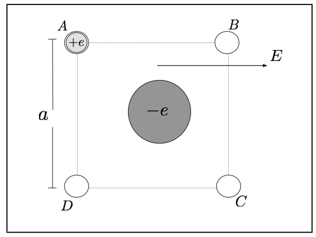
(e) *The mean kinetic energy is*

$$\begin{aligned} \overline{K} &= 3 \times \frac{1}{2} m \overline{v_x^2} \\ &= 3 \times \frac{1}{2} m \int dv_x P(v_x) v_x^2 \\ &= 3 \times \frac{1}{2} m \left(\frac{m\beta}{2\pi}\right)^{1/2} \int dv_x v_x^2 e^{-\beta m v_x^2/2} \\ &= \frac{3}{2\beta} \end{aligned}$$

The mean potential energy is

$$\begin{aligned} \overline{U} &= mg\overline{z} \\ &= mg \int_0^L dz \; zP(z) \\ &= mg \frac{\beta mg}{(1-e^{-\beta mgL})} \int_0^L dzz \; e^{-\beta mgz} \\ &= \frac{1}{\beta}\left[1 - \frac{\beta mgL}{(e^{\beta mgL}-1)}\right] \end{aligned}$$

Problem 3: A two-dimensional solid at temperature T contains N negatively charged impurity ions per unit area, the negative ions replacing some ordinary atoms of the solid. The solid as a whole is electrically neutral, since each negative ion with charge $-e$ has in its vicinity one positive ion with charge $+e$. The positive ion, much smaller, is free to move between each of the four equidistance sites A, B, C and D surrounding the stationary negative ion, as shown. The spacing between the these sites is a and the energy of interaction of the positive ion with the stationary negative ion is $-\epsilon_0$ for each lattice site



(a) What are the relative probabilities of the positive ion being found at the four lattice sites?

(b) The solid is placed in a region of a uniform electric field of magnitude E, as illustrated above. Taking the origin at the location of the negative ion, determine the interaction energy of the system with the external electric field at the four lattice sites (*the interaction energy is* $E_{int} = -\vec{p}\cdot\vec{E}$ *where* $\vec{p}$ *is the dipole moment of the system*).

(c) What now are the relative probabilities of the positive ion being found at the four lattice sites?

(d) The mean polarisation of the solid is the mean dipole moment per unit area along the direction of the electric field. Calculate the polarisation of the solid as a function of temperature and the external electric field E.

(e) Calculate the expression for the polarisation at 'high' temperatures. What temperatures are 'high'?

Solution 3: *The system has four microstates, corresponding to the four possible locations of the positive ion.*

(a) *In absence of an external electric field, each microstate has the same energy. Therefore, they are all equiprobable.*

(b) *The magnitude of the dipole moment of the system corresponding to each position of the positive ion is* $p_0 = ea/\sqrt{2}$. *Given that the interaction energy of a dipole is* $E_{int} = -\vec{p}\cdot\vec{E}$, *the energies of the four microstates are respectively* $\epsilon_A = \epsilon_D = +eaE/2$ *and* $\epsilon_B = \epsilon_C = -eaE/2$.

(c) *The probability of the ion being found at sites* A *and* D *are equal. Similarly, the probability of the ion being found at sites* B *and* C *are equal. The reltive probabilities of the ion being found at sites* A *and* B *is*

$$\begin{aligned}\frac{P_A}{P_B} &= \frac{e^{-\beta\epsilon_A}}{e^{-\beta\epsilon_B}} \\ &= e^{-\beta eaE}\end{aligned}$$

(d) *If the polarisation of the dipole at a lattice site is* $\vec{p}$, *the mean polarisation is the average value of* $p_{\|} = \vec{p}\cdot\hat{E}$, *where* $\hat{E}$ *is a unit vector along* $\vec{E}$. *The value of this quantity at the four lattice sites is* $p_{\|A} = p_{\|D} = -ea/2$ *and* $p_{\|B} = p_{\|C} = +ea/2$. *Given that* $P_A + P_B + P_C + P_D = 1$ *and given the ratio* P_A/P_B, *the probabilities are computed to be*

$$\begin{aligned}P_A &= \frac{1}{2}\left(\frac{1}{1+e^{\beta eaE}}\right) \\ P_B &= \frac{1}{2}\left(\frac{e^{\beta eaE}}{1+e^{\beta eaE}}\right)\end{aligned}$$

Then,

$$\begin{aligned}\overline{p_{\|}} &= 2P_A p_{\|A} + 2P_B p_{\|B} \\ &= \frac{ea}{2}\left(\frac{e^{\beta eaE}-1}{e^{\beta eaE}+1}\right) \\ &= \frac{ea}{2}\tanh\left(\frac{\beta eaE}{2}\right)\end{aligned}$$

Since there are N impurities per unit area, therefore, the mean polarisation is

$$\begin{aligned} \overline{P} &= N\overline{p_{\|}} \\ &= \frac{Nea}{2}\tanh\left(\frac{\beta eaE}{2}\right) \end{aligned}$$

'High' temperatures correspond to the condition $\beta eaE/2 << 1$. Under such conditions,

$$\begin{aligned} \overline{P} &\simeq \frac{Nea}{2}\left(\frac{\beta eaE}{2}\right) \\ &= N\frac{(ea)^2E}{4k_BT} \end{aligned}$$

Problem 4: A sensitive spring balance consists of a quartz spring with spring constant k. This balance is used to measure the mass of very tiny, light objects by suspending them from the balance and observing the extension in the spring. Consider a tiny object of mass m suspended from the spring. The object is in an environment which is at temperature T, and gets 'kicked' around by it, reaching equilibrium with the environment.

1. What is the potential energy of the system if the spring is extended by x?
2. What is the probability that the spring is extended by x relative to its equilibrium length?
3. Calculate the mean extension $\overline{x}$ and the mean squared extension $\overline{(x-\overline{x})^2}$.
4. Comparing the square root of the mean squared extension with the mean extension, estimate the minimum mass that can be reliably measured.

Solution 4:

(a) *If the position of the mass relative to the relaxed position of the spring is x, then its potential energy is*

$$\begin{aligned} U(x) &= \frac{1}{2}kx^2 + mgx \\ &= \frac{1}{2}k(x+x_0)^2 - \frac{m^2g^2}{2k} \end{aligned}$$

where $x_0 = mg/k$.

(b) *The probability of the position of the mass lying between x and $x+dx$ is*

$$P(x)dx = Ne^{-\beta U(x)}dx$$

where N is determined by the normalisation condition $\int_{-\infty}^{\infty} dxP(x) = 1$. The integral is a Gaussian, resulting in

$$P(x)dx = \sqrt{\frac{\beta k}{2\pi}}e^{-\beta k(x+x_0)^2/2}dx$$

(c) *The mean extension is clearly $\overline{x} = -x_0$. The mean square extension is*

$$\begin{aligned} \overline{(x-\overline{x})^2} &= \int_{-\infty}^{\infty} dx(x+x_0)^2P(x) \\ &= \frac{k_BT}{k} \end{aligned}$$

(d) *The fluctuation in position is* $\Delta x = \sqrt{\overline{(x-\overline{x})^2}}$. *For the measurement to be reliable,* $\Delta x << x_0$. *This results in the condition*

$$m >> \frac{1}{g}\sqrt{k_B T k}$$

1.4 Partition Function, Heat Capacity, Entropy

Problem 1: Consider a single particle system with five states. There is one state with energy 0, two states with energy ϵ and two states with energy 2ϵ. The system is in equilibrium with a heat bath at temperature T.

(a) Calculate the partition function for the system.

(b) Calculate the mean energy and heat capacity of the system as functions of temperature.

(c) What is the relative probability of the system having energy 2ϵ and ϵ?

Problem 2: The partition function of a system is given by

$$lnZ = aT^4V$$

where T is the absolute temperature, V is the volume of the system and a is a constant. Evaluate the mean energy, pressure and entropy of the system.

Problem 3: Consider a simplified model of graphite, in which each carbon atom acts as a harmonic oscillator, oscillating with frequency ω within the layer and frequency ω' perpendicular to it. The oscillations in the three directions are independent, such that the expression for energy of a carbon atom is

$$E = \frac{1}{2m}(p_x^2 + p_y^2 + p_z^2) + \frac{m}{2}(\omega^2 x^2 + \omega^2 y^2 + \omega'^2 z^2)$$

where coordinates x, y are in the plane of the layer and z is perpendicular to it. The sample is at temperature T, such that $\hbar\omega >> T$ and $\hbar\omega' << T$ (the restoring forces in the plane of the layer are much stronger than those perpendicular to it).

(a) Given the temperature conditions, one kind of the oscillations (in the plane or perpendicular to it) can be treated classically, and the other quantum mechanically with only the ground and first excited states appreciably populated. Identify the corresponding oscillations.

(b) Taking into account the above considerations, calculate the partition function and show that it factorises into three factors, two of which are identical.

(c) Find an expression for the molar specific heat of the system as a function of temperature, using approximations appropriate to the temperature conditions stated above.

Solution 3:

(a) *Given the temperature conditions, the oscillations perpendicular to the plane can be treated classically, and those in the plane need to be treated quantum mechanically.*

(b) *Since the expression for energy is additive in the contributions along the three independent directions, the partition function for each atom will factorise as*

$$Z = Z_x Z_y Z_z$$

where further, $Z_x = Z_y$. Given that the motion perpendicular to the plane can be treated classically, it follows that

$$
\begin{aligned}
Z_z &= \int dz dp_z e^{-\beta(p_z^2/2m + m\omega'^2 z/2)} \\
&= \frac{2\pi}{\omega'\beta}
\end{aligned}
$$

For the motion in the plane, given that $\hbar\omega >> T$, only the ground and the first excited state contributions are relevant. Then

$$
\begin{aligned}
Z_x &= \sum_{n=0}^{1} e^{-\beta(n+1/2)\hbar\omega} \\
&= e^{-\beta\hbar\omega/2} + e^{-3\beta\hbar\omega/2} \\
&= e^{-\beta\hbar\omega/2}\left(1 + e^{-\beta\hbar\omega}\right)
\end{aligned}
$$

Finally, the partition funciton for a single atom is

$$
\begin{aligned}
Z &= Z_x^2 Z_z \\
&= e^{-\beta\hbar\omega}\left(1 + e^{-\beta\hbar\omega}\right)^2 \left(\frac{2\pi}{\omega'\beta}\right)
\end{aligned}
$$

(c) *The logarithm of the partition function is*

$$
\ln Z = -\beta\hbar\omega + 2\ln\left(1 + e^{-\beta\hbar\omega}\right) - \ln\left(\frac{\beta\omega'}{2\pi}\right)
$$

The mean energy per atom is

$$
\begin{aligned}
\bar{\epsilon} &= -\frac{\partial \ln Z}{\partial \beta} \\
&= \hbar\omega + k_B T + \frac{2\hbar\omega}{1 + e^{\beta\hbar\omega}}
\end{aligned}
$$

The heat capacity per atom is

$$
\begin{aligned}
c &= \frac{\partial \epsilon}{\partial T} \\
&= k_B\left[1 + 2\left(\frac{\hbar\omega}{k_B T}\right)^2 \frac{1}{\left(1 + e^{\hbar\omega/k_B T}\right)^2}\right]
\end{aligned}
$$

Therefore, the molar specific heat of the system will be

$$
C = R\left[1 + 2\left(\frac{\hbar\omega}{k_B T}\right)^2 \frac{1}{\left(1 + e^{\hbar\omega/k_B T}\right)^2}\right]
$$

Problem 4: N diatomic molecules are stuck on a surface. Each molecule can either lie flat on the surface (in which case it can orient itself either along the x or the y direction) or it can stand up perpendicular to the surface (along the z direction). Assume that the flat configurations have zero energy and the configuration perpendicular to the surface has energy $\epsilon > 0$. The system is in thermal equilibrium at temperature $T > 0$.

(a) Calculate the partition function of the system.

(b) Calculate the mean energy of the system. What is the largest possible value for this energy (attained by changing the temperature)?

(c) Calculate the heat capacity of the system as a function of temperature.

(d) What is the probability of a given molecule 'standing up'?

Solution 4: *For each molecule, there are three microstates corresponding to it oriented along the* x, y *or* z *directions. The energies of these microstates are* $\epsilon_x = \epsilon_y = 0$, $\epsilon_z = \epsilon$.

(a) *The partition function for a single molecule is*

$$\begin{aligned} Z_1 &= 1 + 1 + e^{-\beta\epsilon} \\ &= 2 + e^{-\beta\epsilon} \end{aligned}$$

Then, since the molecules are independent, the partition fucntion of the system is

$$\begin{aligned} Z &= Z_1^N \\ &= \left(2 + e^{-\beta\epsilon}\right)^N \end{aligned}$$

(b) *The mean energy of the system is*

$$\begin{aligned} \overline{E} &= -\frac{\partial \ln Z}{\partial \beta} \\ &= \frac{N\epsilon}{1 + 2e^{\beta\epsilon}} \end{aligned}$$

As $T \to \infty, \beta \to 0$. *In this limit,* $\overline{E} \to N\epsilon/3$.

(c) *The heat capacity of the system is*

$$\begin{aligned} C &= \frac{\partial \overline{E}}{\partial T} \\ &= 2Nk_B \left(\frac{\epsilon}{k_B T}\right)^2 \frac{e^{\beta\epsilon}}{1 + 2e^{\beta\epsilon}} \end{aligned}$$

(d) *The probability of a given molecule standing 'up' is*

$$\begin{aligned} P_z &= \frac{1}{Z_1} e^{-\beta\epsilon} \\ &= \frac{1}{1 + 2e^{\beta\epsilon}} \end{aligned}$$

1.5 Negative Temperatures

Problem 1: Consider an isolated system of $N >> 1$ weakly interacting, distinct particles in equilibrium. Each particle can be in one of three states, with energies $0, \epsilon$ and 2ϵ respectively. Given the system has a certain energy, the temperature of the system is given by

$$\begin{aligned} \frac{1}{T} &= \frac{\partial S}{\partial E} \\ &\simeq \frac{\Delta S}{\Delta E} \end{aligned}$$

where ΔS is the change in the entropy of the system due to the change in the energy of the system by ΔE.

(a) Let the entire system be in its ground state. What is its entropy? If the anergy $\Delta E = \epsilon$ is added to the system, what is its entropy? Given the definition of temperature above, what can you say about the temperature of the system if it is in the ground state?

(b) Let the total energy of the system be $2N\epsilon - \epsilon$. What is the entropy of the system? What is the entropy of the system if energy $\Delta E = \epsilon$ is added to it? If the system has energy $2N\epsilon - \epsilon$, what can you say about the temperature of the system?

Solution 1:

(a) *If the system is in its ground state, there is only one microstate corresponding to this (all particles with energy* 0*). Therefore, the entropy of the system is zero. If energy* $\Delta E = \epsilon$ *is added to it, the energy of the system is* ϵ*. This corresponds to* N *possible microstates, in which one particle has energy* ϵ *and all others have energy* 0*. Then, the entropy of the system is* $S = k_B \ln N$*. The temperature of the system in the ground state is positive, since an increase in energy increases entropy.*

(b) *If the total energy of the system is* $2N\epsilon - \epsilon$*, this corresponds to* N *microstates in which one particle has energy* ϵ *and all others have energy* 2ϵ*. The entropy of the system is then* $S = k_B \ln N$*. If energy* ϵ *is added to the system, the number of microstates reduces to one, corresponding to all the particles having energy* 2ϵ*. The entropy therefore reduces to zero. Since addition of energy decreases entropy, the temperature of the system is negative.*

Problem 2: Consider an isolated system of $N >> 1$ weakly interacting, distinct particles in equilibrium. Each particle can be in one of M states with energies $\epsilon_0, 2\epsilon_0, ..., M\epsilon_0$. Can this system exhibit negative temperatures? If so, give a value of energy corresponding to which the temperature of the system is (a) positive (b) negative. (c) If $M \longrightarrow \infty$, will the system exhibit negative temperatures? Give a physical argument.

Solution 2: *The system does exhibit negative temperatures, since the energy of the system is bounded from above, the maximum possible energy being* $NM\epsilon_0$*. (a) The system in the ground state has a positive temperature, since adding energy* ϵ_0 *to the system increases its entropy by* $\Delta S = k_B \ln N$*. (b) The system, when its energy is* $MN\epsilon_0 - \epsilon_0$*, has a negative temperature, since increasing its energy by* ϵ_0 *decreses its entropy by* $\Delta S = -k_B \ln N$*. (c) As* $M \to \infty$*, increasing the energy of the system will always increase its entropy. Therefore, the system will always have a positive temperature.*

Problem 3: Consider two systems A and B, system A consisting of $N_A >> 1$ weakly interacting particles, each of which can be in one of an infinite number of possible quantum states with energies $0, \epsilon, 2\epsilon, 3\epsilon, ...$ System B on the other hand consisting of $N_B >> 1$ weakly interacting particles, each of which can be in one of two quantum states with energies $0, \epsilon$. Initially, these systems are insulated from each other, with system A having total energy $N_A\epsilon$ and B having energy $3N_B\epsilon/4$.

(a) What can you say about the sign of the temperatures of these two systems?

(b) The systems are now made to interact with each other, till they reach equilibrium. What is the sign of the temperature of each system after equilibrium is attained?

Solution 3:

(a) *Since the energy of system A is not bounded from above, it will always have a positive temperature. System B, on the other hand, can exhibit both positive and negative temperatures. It will have a negative temperature for energies greater than $N_B\epsilon/2$, since for energies greater than this, increasing the energy of the system will decrease its entropy. Given that its energy is $3N_B\epsilon/4$, it has a negative temperature.*

(b) *After the systems interact and attain equilibrium, since system A always has a positive temperature, system B will be driven to a positive temperature as well, as a result of losing a part of its energy to system A to maximise the overall entropy.*

1.6 Equipartition Principle

Problem 1: Consider a classical system of $N >> 1$ independent oscillators, each of which has energy given by

$$\epsilon = \frac{p^2}{2m} + \frac{1}{2}kx^2$$

where x and p are the position and momentum of the particle. If the system is in equilibrium at temperature T, what is the molar specific heat of the system? If the expression for energy has a small correction which is not quadratic in x, what will be the qualitative change in the behaviour of the specific heat of the system?

Solution 1: *From the Equipartition Principle, the molar specific heat of the system will be $c_v = R$. If there is a non-quadratic correction, the Equipartition Principle is not applicable, and the molar specific heat will in general depend on the temperature.*

Problem 2: Consider a classical system of $N >> 1$ weakly interacting particles, each of which has energy given by

$$\epsilon = \frac{p^2}{2m} + \frac{1}{2}kx^2 + \lambda x^4$$

where x and p are the position and momentum of the particle. The system is in equilibrium at temperature T.

(a) If $\lambda = 0$, what is the moler specific heat of the system?

(b) If λ is not zero, but the quartic term is very 'small' compared to the quadratic term, what will be the variation in the specific heat of the system with temperature? *Hint: Use $e^u \simeq 1 + u$ for 'small' u.*

Solution 2:

(a) *If $\lambda = 0$, the energy is quadratic in position and momentum. Therefore, the Principle of Equipartition is valid, and the molar specific heat of the system will be $c_v = R$.*

(b) *If $\lambda \neq 0$, the partition function of the system will be*

$$
\begin{aligned}
Z &= \int dxdp\ e^{-\beta(p^2/2m+kx^2/2+\lambda x^4)} \\
&\simeq \int dxdp\ e^{-\beta(p^2/2m+kx^2/2)} \left(1-\beta\lambda x^4\right) \quad \text{since the correction to energy is `small'} \\
&= \frac{2\pi}{\beta}\sqrt{\frac{m}{k}} - \beta\lambda\sqrt{\frac{2m\pi}{\beta}} \times \int_{-\infty}^{\infty} dx\ x^4\ e^{-\beta kx^2/2} \\
&= \frac{2\pi}{\beta}\sqrt{\frac{m}{k}} - \beta\lambda\sqrt{\frac{2m\pi}{\beta}} \times \frac{3\sqrt{\pi}}{4}\left(\frac{2}{\beta k}\right)^{5/2} \\
&= \frac{2\pi}{\beta}\sqrt{\frac{m}{k}}\left(1-\frac{3\lambda}{\beta k^2}\right)
\end{aligned}
$$

Therefore

$$
\begin{aligned}
\ln Z &= \ln Z_0 + \ln\left(1-\frac{3\lambda}{\beta k^2}\right) \\
&\simeq \ln Z_0 - \frac{3\lambda}{\beta k^2} \quad (\because \text{correction is `small'})
\end{aligned}
$$

where Z_0 is the partition function without the correction. The mean energy per particle is

$$
\begin{aligned}
\overline{\epsilon} &= -\frac{\partial \ln Z}{\partial \beta} \\
&= \overline{\epsilon_0} - \frac{3\lambda(k_BT)^2}{k^2}
\end{aligned}
$$

where $\overline{\epsilon_0}$ is the mean energy paer particle if $\lambda = 0$. The heat capacity per particle is

$$
\begin{aligned}
c &= \frac{\partial\overline{\epsilon}}{\partial T} \\
&= c_0 - \frac{6k_B\lambda(k_BT)}{k^2}
\end{aligned}
$$

Therefore, the molar specific heat of the system is

$$
\begin{aligned}
c_v &= N_A \times c \\
&= R - \frac{6R\lambda(k_BT)}{k^2} \\
&= R\left(1-\frac{6\lambda(k_BT)}{k^2}\right)
\end{aligned}
$$

Problem 3: Consider a weakly interacting system of particles, such that the expression for energy of any one particle consists of n terms, quadratic in position and momentum components. If classical physics is an adequate description of this system, what is the molar specific heat of the system? If the temperature of the system is progressively lowered, will the experimentally measured molar specific heat be in agreement with this result? Explain.

Solution 3: *If Classical Physics is an adequate description, then the Principle of Equipartition can be applied, which gives the moolar specific heat to be $c_v = nR/2$. As the temperature is lowered, eventually, quantum effects become significant, and the molar specific heat will be found to be temperature dependent, in disagreement with the classical result.*

2 Theory of Radiation

Problem 1: Estimate the surface temperature of the red giant star Aldebaran, given that it emits radiation with maximum intensity at a wavelength of $7250\AA$. You can use the fact that the maximum intensity of solar radiation is at wavelength $5000\AA$ and corresponds to a surface temperature of about $5780°K$.

Solution 1: *Let the surface temperature of Aldebaran be T_A. The wavelength corresponding to maximum intensity is $\lambda_A = 7250\AA$. The surface temperature of the Sun is $T_S = 5780°K$ and the wavelength corresponding to maximum intensity is $\lambda_S = 5000\AA$. It follows from Wien's Law that $\lambda_A T_A = \lambda_S T_S$. Then,*

$$\begin{aligned} T_A &= \frac{\lambda_S}{\lambda_A} T_S \\ &= \frac{5000}{7250} \times 5780 \\ &= 3986°K \end{aligned}$$

Problem 2: Electromagnetic radiation inside a cavity of volume V is in equilibrium at temperature T. If the temperature of the cavity is halved, by what factor does the pressure change? How does this compare with a classical monoatomic gas under similar conditions?

Solution 2: *The pressure due to thermal radiation is independent of volume, and is proportional to the fourth power of temperature. If the temperature is halved, the pressure will become one-sixteenth. In case of a classical monoatomic gas, the pressure is proportional to temperature, at a given volume. Therefore, if the temperature is halved, the pressure will be reduced to half.*

Problem 3: If Planck's constant were zero, what would be the total energy of radiation contained in a cavity of volume V at temepratur T?

Solution 3: *Putting Planck's constant to zero is equivalent to using the Classical theory of radiation. Therefore, in this limit, the total energy due to thermal radiation would be infinite (ultraviolet catastrophe).*

Problem 4: At what rate does radiation escape from a hole $10cm^2$ in area, in the wall of a furnace whose interior is at a temperature of $1000°K$?

Solution 4: *Assuming the furnace to be a perfect blackbody, the rate at which radiation will escape is*

$$P = A\sigma T^4$$

where σ is Stefan's constant, A is the area of the hole and T is the absolute temperature. Substituting the appropriate values in the expression gives

$$\begin{aligned} P &= 10 \times 10^{-4} \times 5.67 \times 10^{-8} \times (1000)^4 \\ &= 56.7W \end{aligned}$$

Problem 5: Blackbody radiation undergoes adiabatic expansion. Give a combination of thermodynamic quantities that remains unchanged.

Solution 5: *During an adiabatic expansion, the entropy of the system does not change. Therefore, the entropy of radiation will not change. Since $S \propto VT^3$, therefore, the product VT^3 will remain constant.*

Problem 6: The surface temperature of the Sun is about $5500^\circ K$ and its radius about 7×10^8 m. The radius of the Earth is about 6.4×10^4 m and the mean distance of the Earth from the sun is about 1.5×10^{11} m. Assume that the Sun acts as a perfect black body and that the Earth absorbs all the radiation incident on it (and then re-emits it like a blackbody, ignoring greenhouse effects). Given that the Earth is in radiative equilibrium, estimate the temperature of the Earth.

Solution 6: *Energy emitted by Sun per unit time is*

$$P_S = (4\pi R_S^2) \times \sigma T_S^4$$

where R_S is the radius of the Sun and T_S is its temperature. Then intensity of radiation received by the Earth is

$$\begin{aligned} I_E &= (4\pi R_S^2) \times \sigma T_S^4 \times \frac{1}{4\pi R_{SE}^2} \\ &= \left(\frac{R_S}{R_{SE}}\right)^2 \sigma T_S^4 \end{aligned}$$

where R_{SE} si the Sun-Earth distance. Then, total energy incident on the Earth's disc per unit time is

$$P_{abs} = \left(\frac{R_S}{R_{SE}}\right)^2 \sigma T_S^4 \times \pi R_E^2$$

where R_E is the radius of the Earth. The energy emitted per unit time by the Earth is

$$P_{emit} = (4\pi R_E^2) \times \sigma T_E^4$$

where T_E is the temperature of the Earth. In equilibrium, $P_{abs} = P_{emit}$. This gives

$$\begin{aligned} T_E &= T_S \sqrt{\frac{R_S}{2R_{SE}}} \\ &= 5500 \times \sqrt{\frac{7 \times 10^8}{2 \times 1.5 \times 10^{11}}} \\ &\simeq 266^\circ K \end{aligned}$$

Problem 7: The filament of a light bulb is cylindrical with length $l = 20$ mm and radius $r = 0.05$ mm. The filament is maintained at a temperature $T = 5000^\circ K$ by an electric current. The filament behaves approximately as a black body. At night, you observe the light bulb from a distance $D = 10$ km with the pupil of your eye fully dilated to a radius $r_0 = 3$ mm.

(a) What is the total power radiated by the filament?

(b) How much radiative energy per unit time enters your eye?

(c) At what wavelength does the filament radiate the most power?

Solution 7:

(a) *The total power radiated is*

$$P \simeq 2\pi r l \sigma T^4$$

(b) *The amount of radiated energy per unit time entering the eye is*

$$\begin{aligned} P' &= P \times \frac{1}{4\pi D^2} \times \pi r_0^2 \\ &= P\left(\frac{r_0^2}{4D^2}\right) \end{aligned}$$

(c) *Easily computed using Wien's Law.*

Problem 8: A cavity of volume $1cm^3$ is filled with blackbody radiation at temperature $727°K$. What is the average number of photons in the cavity? Use can use the following result:

$$\int_0^\infty dx \frac{x^2}{e^x - 1} = 2.404$$

Solution 8: *The mean occupation number for photons corresponding to mode $\vec{k}$ (with momentum $\vec{p} = \hbar\vec{k}$ is*

$$n_{\vec{k}} = \frac{1}{e^{\beta h \nu_{\vec{k}}} - 1}$$

where $\nu_{\vec{k}} = c\left|\vec{k}\right|$ is the frequency corresponding to the mode. Therefore, the total number of photons is

$$\begin{aligned} N &= \sum_{\vec{k}} n_{\vec{k}} \\ &= \int_0^\infty d\nu g(\nu) \frac{1}{e^{\beta h\nu} - 1} \end{aligned}$$

where $g(\nu)$ is the number of modes in frequency range ν and $\nu + d\nu$ per unit $d\nu$ (density of states), given by

$$g(\nu) = \frac{8\pi V}{c^3}\nu^2$$

where V is the volume of the region enclosing the radiation. substituting, we get

$$N = \frac{8\pi V}{c^3}\int_0^\infty d\nu \frac{\nu^2}{e^{\beta h\nu} - 1}$$

Introducing integration variable $x = \beta h\nu$, we get

$$\begin{aligned} N &= \frac{8\pi V}{c^3}\frac{1}{\beta^3 h^3}\int_0^\infty dx \frac{x^2}{e^x - 1} \\ &= \frac{8\pi V}{c^3} \times (k_B T)^3 \times 2.404 \\ &= 0.244V\left(\frac{k_B T}{\hbar c}\right)^3 \end{aligned}$$

This expression can be used to make the calculation.

Problem 9: Radiation in equilibrium fills a hot enclosure. How high must the temperature of the enclosure be for the radiation pressure to be equal to one atmosphere?

Solution 9: *Use the expression for radiation pressure*

$$P = \frac{4\sigma}{3c}T^4$$

where σ is Stefan's constant.

Problem 10: The cosmic microwave background radiation left over from the Big Bang today fills the universe with blackbody radiation at temperature $T = 2.76°K$. What is the mean number density of photons? Use can use the following result:

$$\int_0^\infty dx \frac{x^2}{e^x - 1} = 2.404$$

Solution 10: *The number density is given by*

$$\frac{N}{V} = 0.244 \left(\frac{k_B T}{\hbar c}\right)^3$$

This expression can be used to make the calculation.

Problem 11: Consider blackbody radiation in equilibrium at temperature T. Let $u(\nu)d\nu$ be the energy density of the radiation in the frequency interval ν to $\nu + d\nu$ and let $\tilde{u}(\lambda)d\lambda$ be the energy density in the wavelength interval λ to $\lambda + d\lambda$. Calculate ν_{max} and λ_{max}, where ν_{max} is the frequency corresponding to which $u(\nu)$ is maximum and λ_{max} the wavelength corresponding to which $\tilde{u}(\lambda)$ is maximum. Are these related as $\lambda_{max} = c\ \nu_{max}$? Why?

Solution 11: *The frequency distribution of energy density is given by*

$$u(\nu)d\nu \quad = \quad \frac{8\pi h}{c^3}\left(\frac{\nu^3}{e^{\beta h\nu} - 1}\right) d\nu$$

Let the wavelength distribution be $\tilde{u}(\lambda)d\lambda$. Given that $\tilde{u}(\lambda)d\lambda = u(\nu)d\nu$, and that $\nu = c/\lambda$, it follows that

$$\tilde{u}(\lambda)d\lambda = \frac{8\pi hc}{\lambda^5}\left(\frac{1}{e^{\beta hc/\lambda} - 1}\right) d\lambda$$

Locating the maximum of the function $u(\nu)$ is equivalent to locating the maximum of the function

$$f(x) = \frac{x^3}{e^x - 1}$$

where $x = \beta h\nu$. The maximum of the function $u(\nu)$ lies at $h\nu_{max} \simeq 2.82 k_B T$ or $\lambda_{max} T = hc/2.82 k_B \simeq 0.35 hc/k_B$. Let us locate the maximum of the function $\tilde{u}(\lambda)$. Setting $\lambda/\beta hc = x$, this is equivalent to locating the maximum of the function

$$g(x) = \frac{1}{x^5}\left(\frac{1}{e^{1/x} - 1}\right)$$

Since the functions are different, it is not surprising that the location of maximum is different. The maximum of $g(x)$ occurs at $x \simeq 0.2$, which gives $\lambda_{max} T \simeq 0.2 hc/k_B$, which is clearly different. The reason for the difference is the different measure chosen, frequency in one case and wavelength in the other. If data is analyzed in terms of frequency, the peak will be observed at one point, and if it is analyzed in terms of wavelength, it will be observed at a different point. This just shows that the constant in Wien's Law appears different depending on what is observed. Therefore, it is not really true to say that Wien's Law implies that maximum emission takes place at a certain wavelength, since this wavelength will turn out to be different depending on how data is analyzed.

Problem 12: A gas cloud in our galaxy emits radiation at a rate of $10^{27}W$. The radiation has maximum intensity at wavelength $\lambda = 10\mu m$. Assuming the cloud to be spherical and that it emits like a blackbody, estimate the diameter of the cloud.

Solution 12: *Let r be the radius of the cloud. Then, it follows from Stefan's Law that the power emitted by the cloud is*

$$P = 4\pi r^2 \sigma T^4$$

The temperature of the cloud can be obtained from Wien's Law, since the peak intensity wavelength is given. Since the power is given, the radius (and therefore diameter) can be determined.

Problem 13: Consider a hypothetical system of massless Bosonic particles which can be emitted and absorbed by matter, just like photons. The system is confined to a two-dimensional area A and is in equilibrium at temperature T. Performing an analysis similar to that for a three-dimensional photon gas, determine the temperature dependence of the energy density (energy per unit area) of the system.

Solution 13: *Visualising the area as a square of sides L, the mode vectors are given by*

$$\vec{k} = \frac{2\pi}{L}\left(l\hat{i} + m\hat{j}\right)$$

where l, m are integers. Let $N(\nu)$ be the number of modes from fequency zero upto frequency ν. This is equal to the number of possible values of (l, m) such that

$$\begin{aligned} \left|\vec{k}\right|^2 &\leq \frac{4\pi^2}{c^2}\nu^2 \\ \implies l^2 + m^2 &\leq \frac{\nu^2}{c^2}L^2 \end{aligned}$$

This equation represents the area of a circle of radius $\nu c/L$. Therefore,

$$\begin{aligned} N(\nu) &= \frac{\pi L^2 \nu^2}{c^2} \\ &= \frac{\pi A \nu^2}{c^2} \end{aligned}$$

Then, the density of states is given by

$$\begin{aligned} g(\nu) &= \frac{dN(\nu)}{d\nu} \\ &= \frac{2\pi A \nu}{c^2} \end{aligned}$$

The energy contained in the frequency range between ν and $\nu + d\nu$ is

$$\begin{aligned} E(\nu)d\nu &= \left(\frac{h\nu}{e^{\beta h\nu} - 1}\right) g(\nu) d\nu \\ &= \frac{2\pi A h}{c^2}\left(\frac{\nu^2}{e^{\beta h\nu} - 1}\right) d\nu \end{aligned}$$

Then, the energy density is

$$\begin{aligned} u(\nu) &= \frac{E(\nu)}{A} \\ &= \frac{2\pi h}{c^2}\left(\frac{\nu^2}{e^{\beta h\nu} - 1}\right) \end{aligned}$$

3 Fermi Dirac and Bose Einstein Statistics

Problem 1: Seven Bosons are arranged in two compartments. The first compartment has 8 cells and the second compartment has 9 cells of equal size. What is the total number of microstates for the macrostate (3,4)?

Solution 1:*Given N compartments, the i^{th} compartment having g_i cells and occupied by n_i Bosons, the number of microstates is given by*

$$\Omega = \Pi_{i=1}^{N} \frac{(n_i + g_i - 1)!}{n_i!(g_i - 1)!}$$

For this problem, $N = 2, g_1 = 8, g_2 = 9, n_1 = 3, n_2 = 4$. Therefore

$$\begin{aligned} \Omega &= \frac{(3+8-1)!}{3!(8-1)!} \times \frac{(4+9-1)!}{4!(9-1)!} \\ &= 59400 \end{aligned}$$

Problem 2: Six Fermions are arranged in two compartments. The first compartment has 7 cells and the second compartment has 8 cells of equal size. What is the total number of microstates for the macrostate (2,4)?

Solution 2:*Given N compartments, the i^{th} compartment having g_i cells and occupied by n_i Fermions, the number of microstates is given by*

$$\Omega = \Pi_{i=1}^{N} \frac{g_i!}{n_i!(g_i - n_i)!}$$

For this problem, $N = 2, g_1 = 7, g_2 = 8, n_1 = 2, n_2 = 4$. Therefore

$$\begin{aligned} \Omega &= \frac{7!}{2!(7-2)!} \times \frac{8!}{4!(8-4)!} \\ &= 1470 \end{aligned}$$

Problem 3: Four weakly interacting particles are confined to a cubical box of volume V, with the energy of any one particle of the form

$$E = \frac{\pi^2 \hbar^2}{2mV^{2/3}} \left(n_x^2 + n_y^2 + n_z^2\right)$$

where n_x, n_y and n_z are natural numbers. What is the energy of the system at absolute zero if the system is (i) Bosonic (ii) Fermionic? Ignore spin.

Solution 3: *At absolute zero, the system is in its ground state. For Bosons, the ground state corresponds to all four particles in the lowest energy state corresponding to $n_x = n_y = n_z = 1$. Then, the energy of the Bosonic system is*

$$\begin{aligned} E_B &= 4 \times \frac{\pi^2 \hbar^2}{2mV^{2/3}} \times 3 \\ &= \frac{6\pi^2 \hbar^2}{mV^{2/3}} \end{aligned}$$

In case of a Fermionic system, only one particle can occupy a given state. The lowest energy state corresponds to $n_x = n_y = n_z = 1$. There are three first excited degenerate but distinct states, *corresponding to* (n_x, n_y, n_z) *being* $(1,1,2),(1,2,1)$ *and* $(2,1,1)$ *respectively. Then, one Fermion will occupy the lowest energy state and three will occupy the three degenerate first excited states. The total ground state energy of the system is then*

$$\begin{aligned} E_F &= \frac{\pi^2\hbar^2}{2mV^{2/3}} \times (3) + 3 \times \frac{\pi^2\hbar^2}{2mV^{2/3}} \times (6) \\ &= \frac{21\pi^2\hbar^2}{2mV^{2/3}} \end{aligned}$$

Problem 4: Consider a system of two weakly interacting particles. Each particle can be in one of two states with energies 0 and ϵ respectively. Calculate the partition function of the system if the system is (i) Bosonic (ii) Fermionic. Calculate the mean energy of the system as a function of temperature and its value as the temperature approaches absolute zero. Give a physical interpretation of the zero temperature result.

Solution 4: *In the occupation number representation, for the Bosonic system, there are three microstates:* $(2,0),(1,1),(0,2)$ *with energies* $0, \epsilon$ *and* 2ϵ *respectively. For the Fermionic syste, there is only one microstate,* $(1,1)$ *with energy* ϵ*. Therefore, the corresponding partition functions are*

$$\begin{aligned} Z_B &= e^{-\beta\times 0} + e^{-\beta\epsilon} + e^{-2\beta\epsilon} \\ &= 1 + e^{-\beta\epsilon} + e^{-2\beta\epsilon} \\ Z_F &= e^{-\beta\epsilon} \end{aligned}$$

The mean energy is given by

$$\begin{aligned} \overline{E}_B &= -\frac{\partial \ln Z_B}{\partial \beta} \\ &= \frac{\epsilon\, e^{-\beta\epsilon} + 2\epsilon\, e^{-2\beta\epsilon}}{1 + e^{-\beta\epsilon} + e^{-2\beta\epsilon}} \\ \overline{E}_F &= -\frac{\partial \ln Z_F}{\partial \beta} \\ &= \epsilon \end{aligned}$$

As $T \to 0, \beta \to \infty$*. Then,* $\overline{E}_B \to 0$ *and* $\overline{E}_F \to \epsilon$*. This is to be expected, as all systems approach the ground state at absolute zero.*

Problem 5: Calculate the Fermi energy for Silver, given that its density is 10.5 g/cc. The atomic mass of Silver is 108 g. Assume there is one free electron per atom.

Problem 6: Given a Fermi gas, what is the mean occupation number for a state with energy $2k_BT$ above the Fermi energy?

Solution 6: *The mean occupation number at temperature* T *for anergy* ϵ *is*

$$n = \frac{1}{e^{\beta(\epsilon-\mu)} + 1}$$

where μ *is the chemical potential, approximately equal to the Fermi energy. It is given that* $\epsilon - \epsilon_F = 2k_BT = 2/\beta$*. Therefore*

$$n = \frac{1}{e^2 + 1}$$

Problem 7: The Fermi energy of free electrons in Silver atoms at $0°K$ is 5.51 eV. What is the average energy per electron?

Solution 7: *The average energy per electron is*

$$\frac{E}{N} = \frac{3}{5}\epsilon_F$$
$$= 3.306\ eV$$

Problem 8: An ideal non-relativistic Fermi gas at absolute zero has Fermi energy ϵ_F, with each particle having mass m. Calculate the mean value of v_x and v_x^2, where v_x is the x component of velocity of a particle.

Solution 8:*From symmetry, it follows that* $\overline{v_x} = 0$. *Furthur, it also follows that* $\overline{v_x^2} = \overline{v_y^2} = \overline{v_z^2} = \overline{v^2}/3$. *Therefore*

$$\overline{v_x^2} = \frac{1}{3}\overline{v^2}$$
$$= \frac{1}{3} \times \frac{2\overline{E}}{Nm}$$

where $\overline{E}/N$ *is the mean energy per particle and* m *is the particle mass. Given that* $\overline{E}/N = 3\epsilon_F/5$, *it follows that*

$$\overline{v_x^2} = \frac{2}{5}\frac{\epsilon_F}{m}$$

Problem 9: Find an expression for the Fermi energy and the average energy per electron at $0°K$ for a free electron gas of N electrons confined to a one-dimensional region of length L.

Solution 9: *The only difference (compared to the three dimensional case) is in the density of states. A particle confined to a one-dimensional region of length* L *has energy states given by*

$$\epsilon_n = \frac{n^2\pi^2\hbar^2}{2mL^2}$$

where n *is a natural number. Let* $N(\epsilon)$ *be the total number of states upto energy* ϵ. *The number of such states is*

$$N(\epsilon) = \sum_n; \quad n \le n_{max} = \left(\frac{2mL^2}{\pi^2\hbar^2}\right)^{1/2}\epsilon^{1/2}$$
$$\simeq \int_0^{n_{max}} dn$$
$$= n_{max}$$
$$= \left(\frac{2mL^2}{\pi^2\hbar^2}\right)^{1/2}\epsilon^{1/2}$$

Therefore, the density of states is

$$g(\epsilon) = \frac{dN(\epsilon)}{d\epsilon}$$
$$= \frac{1}{2}\left(\frac{2mL^2}{\pi^2\hbar^2}\right)^{1/2}\epsilon^{-1/2}$$

At absolute zero, if N is the total number of Fermions, then

$$\begin{aligned} N &= \int_0^{\epsilon_F} d\epsilon g(\epsilon) \\ &= \frac{1}{2}\left(\frac{2mL^2}{\pi^2\hbar^2}\right)^{1/2} \int_0^{\epsilon_F} d\epsilon \epsilon^{-1/2} \\ &= \left(\frac{2mL^2}{\pi^2\hbar^2}\right)^{1/2} \epsilon_F^{1/2} \end{aligned}$$

which gives

$$\epsilon_F = \frac{N^2\pi^2\hbar^2}{2mL^2}$$

The total energy of the system at absolute zero is

$$\begin{aligned} E &= \int_0^{\epsilon_F} d\epsilon g(\epsilon)\epsilon \\ &= \frac{1}{2}\left(\frac{2mL^2}{\pi^2\hbar^2}\right)^{1/2} \int_0^{\epsilon_F} d\epsilon \epsilon^{1/2} \\ &= \frac{1}{3}\left(\frac{2mL^2}{\pi^2\hbar^2}\right)^{1/2} \epsilon_F^{3/2} \\ &= \frac{N}{3}\epsilon_F \end{aligned}$$

Therefore, the average energy per particle equals one-third the Fermi energy.

Problem 10: Consider a system of N Bosons occupying volume V. At high enough temperature T, the system behaves like a classical idea gas, such that the pressure of the system is proportional to T. If the temperature is such that the system is strongly degenerate, given that a certain fraction of atoms is in the ground state (and does not contribute to pressure), what power of temperature do you expect the pressure to be proportionate to? Explain.

Solution 10: Below the B.E. condensation temperature (T_C), the fraction of particles in the ground state is

$$f_0 = 1 - \left(\frac{T}{T_C}\right)^{3/2}$$

These particles do not contribute to pressure, since they have zero momentum. The fraction of particles not in the ground state is

$$f = \left(\frac{T}{T_C}\right)^{3/2}$$

Assuming these particles to contribute to pressure the same way a classical system does, given that the pressure of a classical gas is proportional to temperature T, the pressure exerted by these particles will be approximately

$$\begin{aligned} P &\simeq \frac{Nk_BT}{V} \times \left(\frac{T}{T_C}\right)^{3/2} \\ &\propto T^{5/2} \end{aligned}$$

Problem 11: Consider a free electron gas consisting of N electrons occupying volume V. At high enough temperature T, the system behaves like a classical idea gas, such that the pressure of the system is proportional to T. At absolute zero, the system exerts a non-zero pressure, the Fermi pressure. If the temperature of the system is such that the system is strongly degenerate, given that a certain fraction of electrons are excited above the Fermi energy (and assuming that this fraction exerts pressure just like a classical gas), what power of temperature do you expect the increase in pressure relative to absolute zero to be proportionate to? Explain.

Solution 11: At temperatures such that $T > T_F$, a fraction of about T/T_F of electrons are expected to be excited above the Fermi energy. Assuming these particles to contribute to pressure the same way a classical system does, given that the pressure of a classical gas is proportional to temperature T, the pressure exerted by these particles will be approximately

$$\begin{aligned} P &\simeq \frac{Nk_BT}{V} \times \left(\frac{T}{T_F}\right) \\ &\propto T^2 \end{aligned}$$

Problem 12: Consider an ideal Bose gas in three dimensions with energy-momentum relation $\epsilon \propto p^s$ with $s > 0$. For what range of s will the system undergo a Bose-Einstein condensation at a non-zero temperature?

Solution 12: Assume that particles are confined to a cubical region of side L. Using periodic boundary conditions, the momentum of the particles takes the following discrete values

$$\vec{p} = \frac{2\pi\hbar}{L}\left(n_x\hat{i} + n_y\hat{j} + n_z\hat{k}\right)$$

where n_x, n_y, n_z are integers. Let N be the total number of particles of the system. Given that the energy of a particle is related to the magnitude of momentum $p = |\vec{p}|$ as $\epsilon = kp^s$, we have

$$\begin{aligned} N &= \sum_{n_x,n_y,n_z} \frac{1}{e^{\beta(\epsilon-\mu)} - 1} \\ &= \frac{L^3}{(2\pi\hbar)^3}\int dp_x dp_y dp_z \frac{1}{e^{\beta kp^s - \mu} - 1} \\ &= \frac{4\pi V}{(2\pi\hbar)^3}\int_0^\infty dp\ p^2 \left(\frac{1}{e^{\beta kp^s - \mu} - 1}\right) \\ &= \frac{4\pi V}{(2\pi\hbar)^3}\int_0^\infty dp\ p^2 \left(\frac{\xi e^{-\beta kp^s}}{1 - \xi e^{-\beta kp^s}}\right) \end{aligned}$$

where $\xi = e^{\beta\mu}$. Introducing variable of integration $x = \beta kp^2$, we get

$$N = \frac{4\pi V}{(2\pi\hbar)^3}\frac{1}{s}\left(\frac{k_BT}{k}\right)^{3/s}\int_0^\infty dx\ x^{3/s-1}\left(\frac{\xi e^{-x}}{1 - \xi e^{-x}}\right)$$

If B.E. condensation takes place at temperature T_C, then at this temperature, $\xi = 1$. Then, T_C is given by

$$\begin{aligned} N &= \frac{4\pi V}{(2\pi\hbar)^3}\frac{1}{s}\left(\frac{k_BT_C}{k}\right)^{3/s}\int_0^\infty dx\ x^{3/s-1}\left(\frac{e^{-x}}{1 - e^{-x}}\right) \\ &= \frac{4\pi V}{(2\pi\hbar)^3}\frac{1}{s}\left(\frac{k_BT_C}{k}\right)^{3/s}\sum_{l=1}^{\infty}\int_0^\infty dx\ x^{3/s-1}e^{-lx} \end{aligned}$$

Using integration variable $u = lx$, we get

$$\begin{aligned}
N &= \frac{4\pi V}{(2\pi\hbar)^3}\frac{1}{s}\left(\frac{k_B T_C}{k}\right)^{3/s}\sum_{l=1}^{\infty}\frac{1}{l^{3/s}}\int_0^{\infty} du\; u^{3/s-1}e^{-u} \\
&= \frac{4\pi V}{(2\pi\hbar)^3}\frac{1}{s}\left(\frac{k_B T_C}{k}\right)^{3/s}\sum_{l=1}^{\infty}\frac{1}{l^{3/s}} \\
&= \frac{4\pi V}{(2\pi\hbar)^3}\frac{1}{s}\left(\frac{k_B T_C}{k}\right)^{3/s}\Gamma(3/s)\sum_{l=1}^{\infty}\frac{1}{l^{3/s}} \\
&= \frac{4\pi V}{(2\pi\hbar)^3}\frac{1}{s}\left(\frac{k_B T_C}{k}\right)^{3/s}\Gamma(3/s)\;\zeta(3/s)
\end{aligned}$$

where $\zeta(3/s)$ is the Riemann-Zeta function, which converges when $3/s > 1$, or for $s < 3$. Therefore, a non-zero T_C will exist for $0 < s < 3$. this is the range of s for which B.E. condensation will take place.

Problem 13: Consider a system of non-interacting quantum particles in three dimensions with dispersion relation $\epsilon \propto k^s$ where ϵ is energy and $\vec{k}$ is the wave-vector, where s is an integer. To what power of ϵ is the density of states proportional to?

Solution 13: Let the energy of a partilce be of the form $\epsilon = \alpha k^s$ where α is a constant. Assume that particles are confined to a cubical region of side L. Using periodic boundary conditions, the wave-vector of the particles takes the following discrete values

$$\vec{k} = \frac{2\pi}{L}\left(n_x\hat{i} + n_y\hat{j} + n_z\hat{k}\right)$$

where n_x, n_y, n_z are integers. Let the number of states upto energy ϵ be $N(\epsilon)$. This is given by

$$\begin{aligned}
N(\epsilon) &= \int dn_x dn_y dn_z \\
&= \frac{V}{(2\pi)^3}\int dk_x dk_y dk_z \\
&= \frac{4\pi V}{(2\pi)^3}\int dk\; k^2 \\
&= \frac{4\pi V}{3}\frac{1}{(2\pi)^3}k^3 \\
&= \frac{4\pi V}{3}\frac{1}{(2\pi)^3}\left(\frac{\epsilon}{\alpha}\right)^{3/s}
\end{aligned}$$

The density of states is given by

$$\begin{aligned}
g(\epsilon) &= \frac{dN(\epsilon)}{d\epsilon} \\
&= \frac{4\pi V}{s}\frac{1}{(2\pi)^3}\left(\frac{1}{\alpha}\right)^{3/s}\epsilon^{(3/s-1)}
\end{aligned}$$

Problem 14: Consider a system of N weakly interacting non-relativistic Bosons confined to a two-dimensional region of area A. Repeating the standard analysis in three dimensions, test whether Bose-Einstein condensation occurs in this system at a non-zero temperature.

Solution 14: *The key idea is to test whether the chemical potential of the system is zero at a non-zero temperature. First, we compute the density of states in two-dimensions. The allowed values of energy for* *a particle confined to a two-dimensional area A are*

$$\epsilon = \frac{\pi^2\hbar^2}{2mA}(n_x^2 + n_y^2)$$

where n_x, n_y are natural numbers. Let $N(\epsilon)$ be the number of states upto energy ϵ. This is given by

$$N(\epsilon) = \int dn_x dn_y$$

subject to the condition

$$n_x^2 + n_y^2 \leq \frac{2mA}{\pi^2\hbar^2}\ \epsilon$$

Clearly, $N(\epsilon)$ is equal to the area of a circle with radius R such that $R^2 = 2mA\epsilon/\pi^2\hbar^2$. Then

$$N(\epsilon) = \frac{2m\pi A}{\pi^2\hbar^2}\ \epsilon$$

The density of states is given by

$$\begin{aligned} g(\epsilon) &= \frac{dN(\epsilon)}{d\epsilon} \\ &= \frac{2m\pi A}{\pi^2\hbar^2} \end{aligned}$$

Let the total number of particles of the system be N. Then

$$\begin{aligned} N &= \int_0^\infty d\epsilon\ g(\epsilon)\frac{1}{e^{\beta(\epsilon-\mu)} - 1} \\ &= \frac{2m\pi A}{\pi^2\hbar^2}\int_0^\infty d\epsilon\frac{\xi e^{-\beta\epsilon}}{1 - \xi e^{-\beta\epsilon}} \end{aligned}$$

where $\xi = e^{\beta\mu}$. Changing integration variable from ϵ to $x = \beta\epsilon$ gives

$$N = \frac{2m\pi A}{\pi^2\hbar^2}(k_B T)\int_0^\infty dx\frac{\xi e^{-x}}{1 - \xi e^{-x}}$$

B.E. condensation will take place if for a finite $T = T_C \neq 0$, $xi = 1$ (or $\mu = 0$). The above equation allows us to compute T_C

$$N = N = \frac{2m\pi A}{\pi^2\hbar^2}(k_B T_C)\int_0^\infty dx\frac{e^{-x}}{1 - e^{-x}}$$

The integral is divergent, so that $T_C = 0$, which shows that B.E. condensation cannot take place in a two-dimensional system of weakly interacting particles.