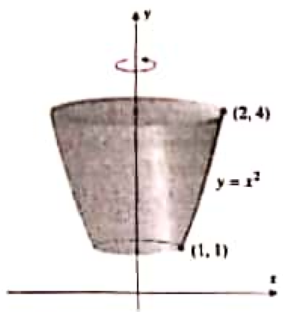


Figure 6.5.7

Example 2 Find the area of the surface that is generated by revolving the portion of the curve $y = x^2$ between $x = 1$ and $x = 2$ about the y-axis (Figure 6.5.7).

Solution. Because the curve is revolved about the y-axis we will apply Formula (5). Toward this end, we rewrite $y = x^2$ as $x = \sqrt{y}$ and observe that the y-values corresponding to $x = 1$ and $x = 2$ are $y = 1$ and $y = 4$. Since $x = \sqrt{y}$, we have $dx/dy = 1/(2\sqrt{y})$, and hence from (5) the surface area S is

$$\begin{aligned} S &= \int_1^4 2\pi x\sqrt{1+\left(\frac{dx}{dy}\right)^2}\,dy \\ &= \int_1^4 2\pi\sqrt{y}\sqrt{1+\left(\frac{1}{2\sqrt{y}}\right)^2}\,dy \\ &= \pi\int_1^4 \sqrt{4y+1}\,dy \\ &= \frac{\pi}{4}\int_5^{17} u^{1/2}\,du \qquad \boxed{\begin{matrix} u = 4y+1 \\ du = 4\,dy \end{matrix}} \\ &= \frac{\pi}{4}\cdot\frac{2}{3}u^{3/2}\Big]_{u=5}^{17} = \frac{\pi}{6}(17^{3/2} - 5^{3/2}) \approx 30.85 \end{aligned}$$

◀

Exercise Set 6.5 ▣ CAS

In Exercises 1–4, find the area of the surface generated by revolving the given curve about the x-axis.

1. $y = 7x,\ 0 \le x \le 1$
2. $y = \sqrt{x},\ 1 \le x \le 4$
3. $y = \sqrt{4 - x^2},\ -1 \le x \le 1$
4. $x = \sqrt[3]{y},\ 1 \le y \le 8$

In Exercises 5–8, find the area of the surface generated by revolving the given curve about the y-axis.

5. $x = 9y + 1,\ 0 \le y \le 2$
6. $x = y^3,\ 0 \le y \le 1$
7. $x = \sqrt{9 - y^2},\ -2 \le y \le 2$
8. $x = 2\sqrt{1 - y},\ -1 \le y \le 0$

In Exercises 9–12, use a CAS to find the exact area of the surface generated by revolving the curve about the stated axis.

▣ 9. $y = \sqrt{x} - \frac{1}{3}x^{3/2},\ 1 \le x \le 3;$ x-axis

▣ 10. $y = \frac{1}{3}x^3 + \frac{1}{4}x^{-1},\ 1 \le x \le 2;$ x-axis

▣ 11. $8xy^2 = 2y^6 + 1,\ 1 \le y \le 2;$ y-axis

▣ 12. $x = \sqrt{16 - y},\ 0 \le y \le 15;$ y-axis

In Exercises 13 and 14, use a CAS or a calculator with numerical integration capabilities to approximate the area of the surface generated by revolving the curve about the stated axis. Round your answer to two decimal places.

▣ 13. $y = \sin x,\ 0 \le x \le \pi;$ x-axis

▣ 14. $x = \tan y,\ 0 \le y \le \pi/4;$ y-axis

15. Use Formula (4) to show that the lateral area S of a right circular cone with height h and base radius r is
$$S = \pi r\sqrt{r^2 + h^2}$$

16. Show that the area of the surface of a sphere of radius r is $4\pi r^2$. [*Hint:* Revolve the semicircle $y = \sqrt{r^2 - x^2}$ about the x-axis.]

17. (a) The figure in Exercise 37 of Section 6.2 shows a spherical cap of height h cut from a sphere of radius r. Show that the surface area S of the cap is $S = 2\pi rh$. [*Hint:* Revolve an appropriate portion of the circle $x^2 + y^2 = r^2$ about the y-axis.]

    (b) The portion of a sphere that is cut by two parallel planes is called a ***zone***. Use the result in part (a) to show that the surface area of a zone depends on the radius of the sphere and the distance between the planes, but not on the location of the zone.

Exercises 18–24 require the formulas developed in the following discussion: If $x'(t)$ and $y'(t)$ are continuous functions and if no segment of the curve

$$x = x(t), \quad y = y(t) \qquad (a \le t \le b)$$

is traced more than once, then it can be shown that the area of the surface generated by revolving this curve about the x-axis is

$$S = \int_a^b 2\pi y(t)\sqrt{[x'(t)]^2 + [y'(t)]^2}\, dt \qquad \text{(A)}$$

and the area of the surface generated by revolving the curve about the y-axis is

$$S = \int_a^b 2\pi x(t)\sqrt{[x'(t)]^2 + [y'(t)]^2}\, dt \qquad \text{(B)}$$

18. Derive Formulas (4) and (5) from Formulas (A) and (B) above by choosing appropriate parametrizations for the curves $y = f(x)$ and $x = g(y)$.

19. Find the area of the surface generated by revolving the parametric curve $x = t^2$, $y = 2t$, $0 \le t \le 4$ about the x-axis.

20. Use a CAS to find the area of the surface generated by revolving the parametric curve

$$x = \cos^2 t, \quad y = 5\sin t \qquad (0 \le t \le \pi/2)$$

about the x-axis.

21. Find the area of the surface generated by revolving the parametric curve $x = t$, $y = 2t^2$, $0 \le t \le 1$ about the y-axis.

22. Find the area of the surface generated by revolving the parametric curve $x = \cos^2 t$, $y = \sin^2 t$, $0 \le t \le \pi/2$ about the y-axis.

23. By revolving the semicircle

$$x = r\cos t, \quad y = r\sin t \qquad (0 \le t \le \pi)$$

about the x-axis, show that the surface area of a sphere of radius r is $4\pi r^2$.

24. The equations

$$x = a\phi - a\sin\phi, \quad y = a - a\cos\phi \qquad (0 \le \phi \le 2\pi)$$

represent one arch of a cycloid. Show that the surface area generated by revolving this curve about the x-axis is $S = 64\pi a^2/3$. [*Hint:* Use the identities $\sin^2\frac{\phi}{2} = \frac{1-\cos\phi}{2}$ and $\sin^3\phi = (1-\cos^2\phi)\sin\phi$ to help with the integration.]

25. (a) If a cone of slant height l and base radius r is cut along a lateral edge and laid flat, then as shown in the accompanying figure it becomes a sector of a circle of radius l. Use the formula $A = \frac{1}{2}l^2\theta$ for the area of a sector with radius l and central angle θ (in radians) to show that the lateral surface area of the cone is $\pi r l$.
(b) Use the result in part (a) to obtain Formula (1) for the lateral surface area of a frustum.

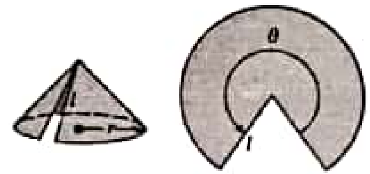


Figure Ex-25

26. Assume that $y = f(x)$ is a smooth curve on the interval $[a, b]$ and assume that $f(x) \ge 0$ for $a \le x \le b$. Derive a formula for the surface area generated when the curve $y = f(x)$, $a \le x \le b$, is revolved about the line $y = -k$ $(k > 0)$.

27. Let $y = f(x)$ be a smooth curve on the interval $[a, b]$ and assume that $f(x) \ge 0$ for $a \le x \le b$. By the Extreme-Value Theorem (4.5.3), the function f has a maximum value K and a minimum value k on $[a, b]$. Prove: If L is the arc length of the curve $y = f(x)$ between $x = a$ and $x = b$ and if S is the area of the surface that is generated by revolving this curve about the x-axis, then

$$2\pi k L \le S \le 2\pi K L$$

28. Let $y = f(x)$ be a smooth curve on $[a, b]$ and assume that $f(x) \ge 0$ for $a \le x \le b$. Let A be the area under the curve $y = f(x)$ between $x = a$ and $x = b$ and let S be the area of the surface obtained when this section of curve is revolved about the x-axis.
(a) Prove that $2\pi A \le S$.
(b) For what functions f is $2\pi A = S$?

6.6 WORK

$$S = \int_a^b 2\pi f(x)\sqrt{1 + [f'(x)]^2}\,dx = \int_a^b 2\pi y\sqrt{1 + \left(\frac{dy}{dx}\right)^2}\,dx \quad (4)$$

Moreover, if g is nonnegative and $x = g(y)$ is a smooth curve on the interval $[c, d]$, then the area of the surface that is generated by revolving the portion of a curve $x = g(y)$ between $y = c$ and $y = d$ about the y-axis can be expressed as

(1, 1)

$$S = \int_c^d 2\pi g(y)\sqrt{1 + [g'(y)]^2}\,dy = \int_c^d 2\pi x\sqrt{1 + \left(\frac{dx}{dy}\right)^2}\,dy \quad (5)$$

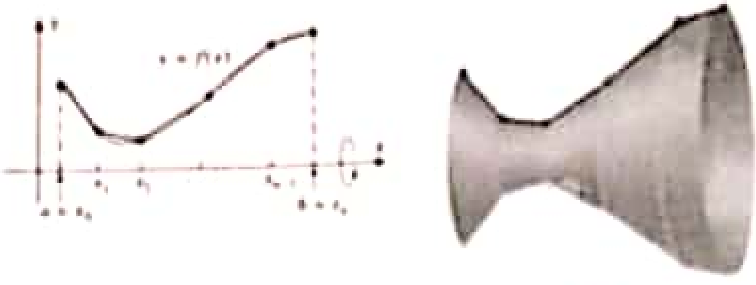


Figure 6.5.3 (a) (b)

To motivate an appropriate definition for the area S of a surface of revolution, we will decompose the surface into small sections whose areas can be approximated by elementary formulas, add the approximations of the areas of the sections to form a Riemann sum that approximates S, and then take the limit of the Riemann sums to obtain an integral for the exact value of S.

To implement this idea, divide the interval $[a, b]$ into n subintervals by inserting numbers $x_1, x_2, \ldots, x_{n-1}$ between $a = x_0$ and $b = x_n$. As illustrated in Figure 6.5.3a, the corresponding points on the graph of f define a polygonal path that approximates the curve $y = f(x)$ over the interval $[a, b]$. When this polygonal path is revolved about the x-axis, it generates a surface consisting of n parts, each of which is a frustum of a right circular cone (Figure 6.5.3b). Thus, the area of each part of the approximating surface can be obtained from the formula

Figure 6.5.4

$$S = \pi(r_1 + r_2)l \tag{1}$$

for the lateral area S of a frustum of slant height l and base radii r_1 and r_2 (Figure 6.5.4). As suggested by Figure 6.5.5, the kth frustum has radii $f(x_{k-1})$ and $f(x_k)$ and height Δx_k. Its slant height is the length L_k of the kth line segment in the polygonal path, which from Formula (1) of Section 6.4 is

$$L_k = \sqrt{(\Delta x_k)^2 + [f(x_k) - f(x_{k-1})]^2}$$

Thus, the lateral area S_k of the kth frustum is

$$S_k = \pi[f(x_{k-1}) + f(x_k)]\sqrt{(\Delta x_k)^2 + [f(x_k) - f(x_{k-1})]^2}$$

If we add these areas, we obtain the following approximation to the area S of the entire surface:

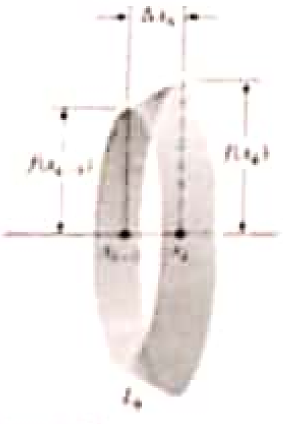


Figure 6.5.5

$$S \approx \sum_{k=1}^{n} \pi[f(x_{k-1}) + f(x_k)]\sqrt{(\Delta x_k)^2 + [f(x_k) - f(x_{k-1})]^2} \tag{2}$$

To put this in the form of a Riemann sum we will apply the Mean-Value Theorem (4.8.2). This theorem implies that there is a number x_k^* between x_{k-1} and x_k such that

$$\frac{f(x_k) - f(x_{k-1})}{x_k - x_{k-1}} = f'(x_k^*) \quad \text{or} \quad f(x_k) - f(x_{k-1}) = f'(x_k^*)\Delta x_k$$

and hence we can rewrite (2) as

$$S \approx \sum_{k=1}^{n} \pi[f(x_{k-1}) + f(x_k)]\sqrt{1 + [f'(x_k^*)]^2}\,\Delta x_k \tag{3}$$

However, this is not yet a Riemann sum because it involves the variables x_{k-1} and x_k. To eliminate these variables from the expression, observe that the average value of the numbers $f(x_{k-1})$ and $f(x_k)$ lies between these numbers, so the continuity of f and the Intermediate-Value Theorem (2.5.8) imply that there is a number x_k^{**} between x_{k-1} and x_k such that

$$\tfrac{1}{2}[f(x_{k-1}) + f(x_k)] = f(x_k^{**})$$

SURFACE AREA

A ***surface of revolution*** is a surface that is generated by revolving a plane curve about an axis that lies in the same plane as the curve. For example, the surface of a sphere can be generated by revolving a semicircle about its diameter, and the lateral surface of a right circular cylinder can be generated by revolving a line segment about an axis that is parallel to it (Figure 6.5.1).

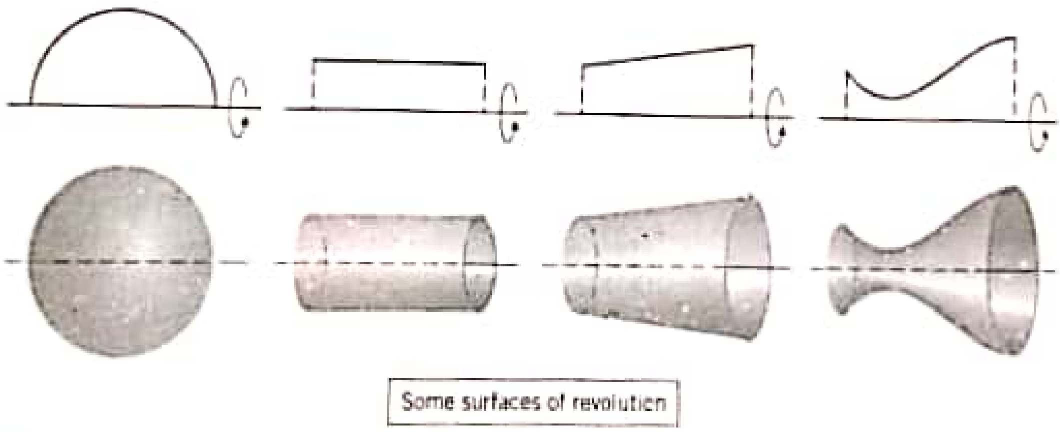


Figure 6.5.1

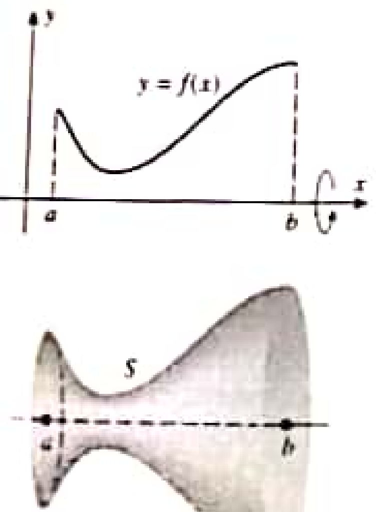


Figure 6.5.2

In this section we will be concerned with the following problem:

> **6.5.1** SURFACE AREA PROBLEM. Suppose that f is a smooth, nonnegative function on $[a, b]$ and that a surface of revolution is generated by revolving the portion of the curve $y = f(x)$ between $x = a$ and $x = b$ about the x-axis (Figure 6.5.2). Define what is meant by the *area* S of the surface, and find a formula for computing it.

(9)

Meeting / Appointment / Planning

(Ex2) Surface area about y-axis

$x = 9y+1 \qquad 0 \le y \le 2$

Sol^n

$$S = \int_c^d 2\pi\, g(y) \sqrt{1 + g'(y)^2}\, dy$$

here $g(y) = 9y+1$, $c = 0$ $d = 2$

$g'(y) = 9$

$1 + g'(y)^2 = 1 + 81 = 82$

$$S = \int_0^2 2\pi\,(9y+1)\sqrt{82}\, dy$$

$$S = 2\pi\sqrt{82} \int_0^2 (9y+1)\, dy$$

$$S = 2\sqrt{82}\,\pi \left(\frac{9y^2}{2} + y \right) \Big|_0^2$$

$$= 2\sqrt{82}\,\pi \left(\frac{9 \times 4}{2} + 2 - 0 \right)$$

$$= 40\sqrt{82}\,\pi$$

Important works to do

JUNE 2011

Wk	M	T	W	T	F	S	S
22			1	2	3	4	5
23	6	7	8	9	10	11	12
24	13	14	15	16	17	18	19
25	20	21	22	23	24	25	26
26	27	28	29	30			

JULY 2011

Wk	M	T	W	T	F	S	S
26					1	2	3
27	4	5	6	7	8	9	10
28	11	12	13	14	15	16	17
29	18	19	20	21	22	23	24
30	25	26	27	28	29	30	31

(3)

Meeting / Appointment / Planning

Surface area formula

$$S = \int_a^b 2\pi f(x)\sqrt{1+f'(x)^2}\,dx$$

here $a = 0$, $b = 1$

$f(x) = x^3$

$f'(x) = 3x^2$.

$$S = \int_0^1 2\pi \cdot x^3 \sqrt{1+9x^4}\,dx$$

let $u = 1 + 9x^4$

$du = 9 \times 4x^3\,dx = 36x^3\,dx \Rightarrow x^3\,dx = \frac{du}{36}$

limits changes from $x = 0 \Rightarrow u = 1$
$x = 1 \Rightarrow u = 10$

Important works to do

$$S = \int_1^{10} 2\pi\sqrt{u}\cdot\frac{du}{36}$$

$$= \frac{2\pi}{36}\frac{u^{3/2}}{3/2}\Big|_1^{10} = \frac{4\pi}{36\times3}(\sqrt{10}-1)$$

$$= \frac{\pi}{27}(\sqrt{10}-1)$$

JUNE 2011

Wk	M	T	W	T	F	S	S
22			1	2	3	4	5
23	6	7	8	9	10	11	12
24	13	14	15	16	17	18	19
25	20	21	22	23	24	25	26
26	27	28	29	30			

JULY 2011

Wk	M	T	W	T	F	S	S
26					1	2	3
27	4	5	6	7	8	9	10
28	11	12	13	14	15	16	17
29	18	19	20	21	22	23	24
30	25	26	27	28	29	30	31

(2)

If the revolution is about y-axis, then surface area from $y=c$ to $y=d$ is given by for $x = g(y)$

$$S = \int_c^d 2\pi\, g(y) \sqrt{1 + g'(y)^2}\, dy$$

Note $y = f(x)$

so $f'(x) = \frac{dy}{dx}$

and $x = g(y)$

so $g'(y) = \frac{dx}{dy}$.

Examples Find the area of the surface generated by revolving the curve $y = x^3$ between $x=0$ and $x=1$ about y-axis

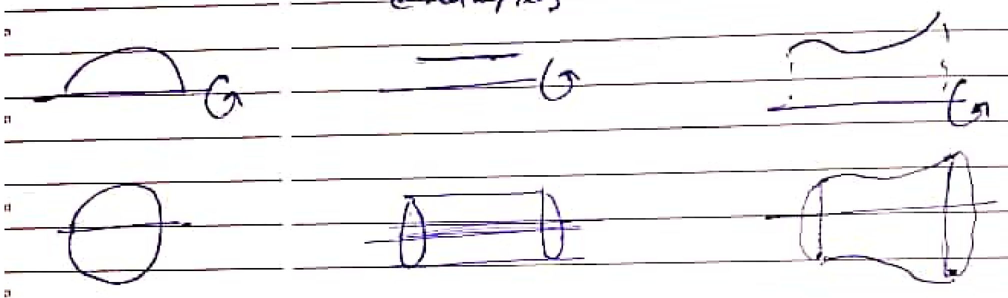
(1)

Seema Gupta 05/04/2020

Area of a Surface of Revolution

A surface of revolution is a surface generated by revolving a plane curve about an axis that lies in the same plane as the curve.

examples



surface Area formula

curve $y = f(x)$ defined on $[a, b]$ revolved about x-axis then surface Area from a to b is given by

$$S = \int_a^b 2\pi f(x) \sqrt{1 + f'(x)^2}\, dx$$