Euler's and Modified Euler's Method for solving Initial Value Problems

FOR THE CLASS OF BSC(PHYSICAL SCIENCES) VI SEMESTER

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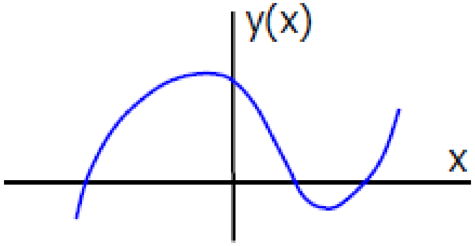
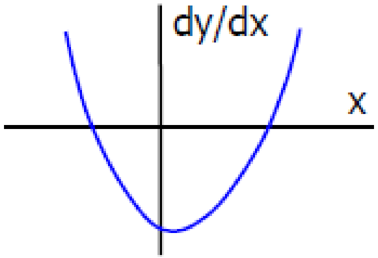
Mathematical Background

- Consider the function $y = x^3 - x^2 - 4x + 4$



- An ODE can be formed by taking dy/dx

$dy/dx = 3x^2 - 2x - 4$



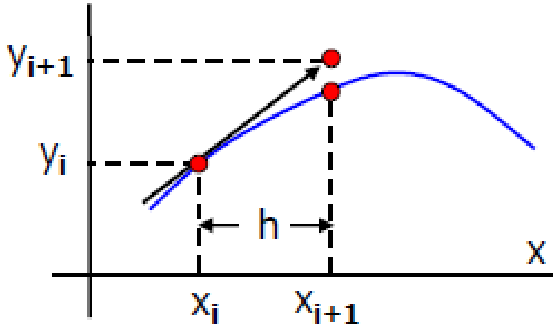
- They both describe the same behavior. One directly shows the dependent variable. The other shows the rate of change of it.
- Solving an ODE is actually going back from dy/dx to y(x). This can be done by integration.

$y(x) = x^3 - x^2 - 4x + C$

- C is the integration constant. For different integration constants different equations will be obtained (If the ODE was 2nd order than we would integrate twice and got two constants).
- Only C=4 corresponds to the original function.
- C can be determined if y(x) is known at one point. This value is known as the **INITIAL VALUE**.
- For example if the **INITIAL CONDITION** $y(0)=4$ is known, than C can be determined.
- A typical numerical solution of an ODE starts from the initial value and discretely constructs y(x) using the differential equation.

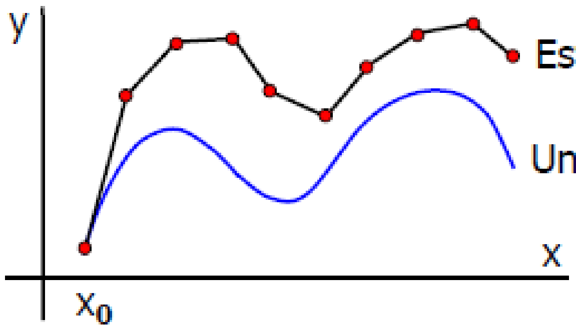
Euler's Method

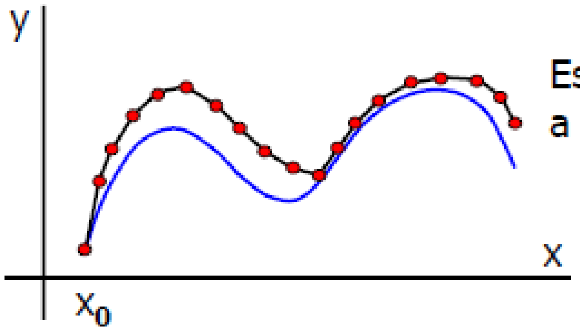
- General form of the ODEs that we will study is $\frac{dy}{dx} = f(x,y)$
- f(x,y) is known and y(x) is to be determined.



$$y_{i+1} = y_i + \left.\frac{dy}{dx}\right|_i h \quad \rightarrow \quad \boxed{y_{i+1} = y_i + f(x_i, y_i)\,h}$$

- Start from the initial condition x_0. Choose a step size h and apply Euler's method successively.





Example [redacted]

- Solve the following initial value problem over the interval [0,2] using (a) h=0.5, (b) h=0.25.

  Note that the analytical solution is $y(x) = e^{x^3/3 - 1.2x}$

$$\frac{dy}{dx} = yx^2 - 1.2y \qquad y(0) = 1$$

(a) $h = 0.5$. $y_{i+1} = y_i + f(x_i, y_i)\,h$ where $f(x) = yx^2 - 1.2y$

$i = 0$ $\quad y_1 = y_0 + f(x_0, y_0)\,h \rightarrow y_1 = y(0.5) = 1 + (1*0^2 - 1.2*1)\,0.5 = 0.4$

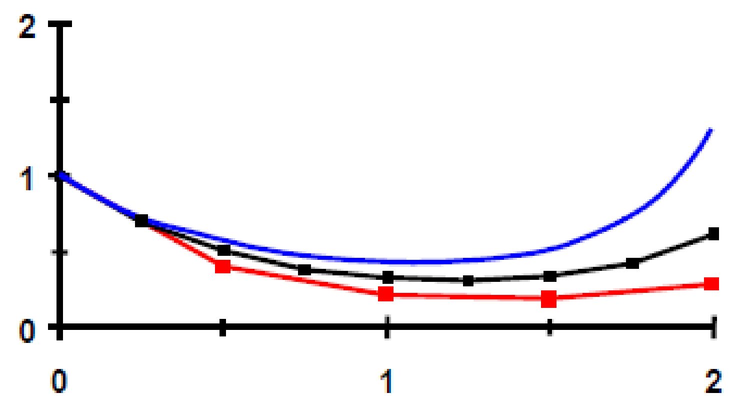
$i = 1$ $\quad y_2 = y_1 + f(x_1, y_1)\,h \rightarrow y_2 = y(1.0) = 0.4 + (0.4*0.5^2 - 1.2*0.4)\,0.5 = 0.21$

$i = 2$ $\quad y_3 = y_2 + f(x_2, y_2)\,h \rightarrow y_3 = y(1.5) = 0.21 + (0.21*1^2 - 1.2*0.21)\,0.5 = 0.189$

$i = 3$ $\quad y_4 = y_3 + f(x_3, y_3)\,h \rightarrow y_4 = y(2.0) = 0.189 + (0.189*1.5^2 - 1.2*0.189)\,0.5 = 0.28823$

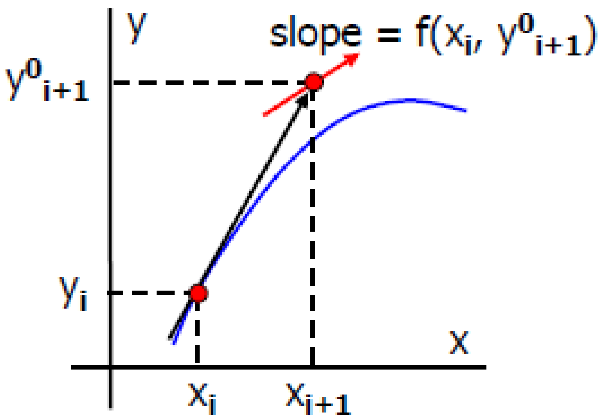
- Analytical solution is shown with blue.
- Solution for h=0.5 is shown with red.

Exercise [redacted] Solve the above example for h=0.25 to obtain the solution shown in the middle (with black).



Heun's Method

- Euler's Method: $y_{i+1} = y_i + f(x_i, y_i)\, h$.
- Can we use a better estimate for the derivative instead of $f(x_i,y_i)$.

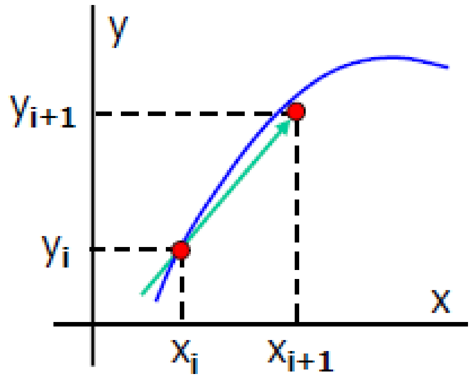


Predictor step:

- Use Euler's Method to find a first estimate for y_{i+1}.

$$y^0_{i+1} = y_i + f(x_i, y_i)\, h$$

- Using y^0_{i+1} calculate the slope at x_{i+1}.



Corrector step:

- Take the average of slopes at x_i and x_{i+1}.
- Use it to calculate a new estimate for y_{i+1}.

$$y_{i+1} = y_i + \frac{f(x_i, y_i) + f(x_{i+1}, y^0_{i+1})}{2}\, h$$

Heun's Method (cont'd)

- Heun's Method is a predictor-corrector method.

- Corrector step can be used more than once to get better estimates for y_{i+1}.

  Predictor: $y^0_{i+1} = y_i + f(x_i, y_i)\, h$

  Corrector: $y^1_{i+1} = y_i + [\, f(x_i, y_i) + f(x_{i+1}, y^0_{i+1})]/2 * h$

  Corrector: $y^2_{i+1} = y_i + [\, f(x_i, y_i) + f(x_{i+1}, y^1_{i+1})]/2 * h$

  ... continue until the error falls below the tolerance

- If $f=f(x)$ only, than the predictor step is not required. Corrector step becomes

$$y_{i+1} = y_i + \frac{f(x_i) + f(x_{i+1})}{2} h$$

  Note the similarity between the above formula and the trapezoidal Rule.

- Heun's Method is 2nd order accurate. It can obtain exact results when the solution $y(x)$ is quadratic.

- It has a global error of $O(h^2)$ and local error of $O(h^3)$.

Example

Solve $\mathbf{dy/dx = yx^2 - 1.2y}$ with the initial condition $\mathbf{y(0)=1}$ over the interval [0,2] using Heun's method. Use h= 0.5. Iterate the corrector to ε_s = 1%. Analytical solution is $y(x) = e^{x^3/3 - 1.2x}$

$h = 0.5$, $f(x) = yx^2 - 1.2y$, $y^0_{i+1} = y_i + f(x_i, y_i)\, h$, $y^k_{i+1} = y_i + [f(x_i, y_i) + f(x_{i+1}, y^{k-1}_{i+1})]/2 * h$

$i = 0$ Predictor: $y^0_1 = 1 + (1*0^2 - 1.2*1)\, 0.5 = 0.4$

Corrector:

k=1 $y^1_1 = 1 + [(1*0^2 - 1.2*1) + (0.4*0.5^2 - 1.2*0.4)]/2 * 0.5 = 0.605$ ε_a= 33.9 %

k=2 $y^2_1 = 1 + [(1*0^2 - 1.2*1) + (0.605*0.5^2 - 1.2*0.605)]/2 * 0.5 = 0.5563125$ ε_a= 8.75 %

k=3 $y^3_1 = 1 + [(1*0^2 - 1.2*1) + (0.5563*0.5^2 - 1.2*0.5563)]/2 * 0.5 = 0.5678757$ ε_a= 2.04 %

k=4 $y^4_1 = 1 + [(1*0^2 - 1.2*1) + (0.5679*0.5^2 - 1.2*0.5679)]/2 * 0.5 = 0.5651295$ ε_a= 0.49 %

$y_1 = y(0.5) = 0.5651295$

$i = 1$ Predictor: $y^0_2 = 0.5651295 + (0.5651295*0.5^2 - 1.2*0.5651295)\, 0.5 = 0.2966930$

Corrector:

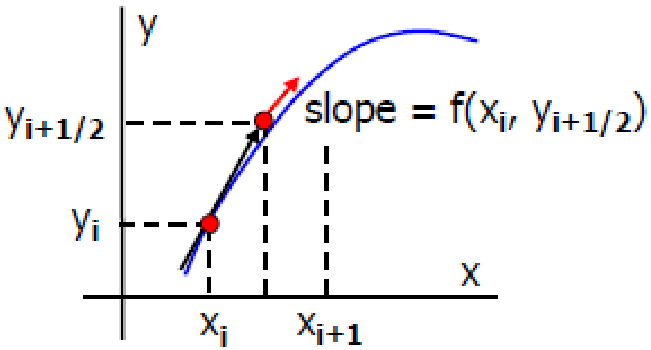
k=1 $y^1_2 = 0.5651295 + [(05651295*0.5^2 - 1.2*05651295) + (0.296693*1^2 - 1.2*0.296693)]/2*0.5$
$= 0.4160766$ ε_a= 28.7 %

.....

Continue like this to find y(1)=0.4104059, y(1.5)=0.5279021, y(2)=2.181574

Midpoint Method

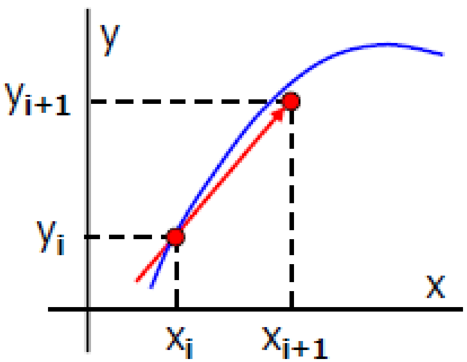
- Similar to Heun's Method this also tries to improve the Euler's Method by using a better slope.



- Use Euler's Method to find an estimate for $y_{i+1/2}$.

$$\mathbf{y_{i+1/2} = y_i + f(x_i, y_i)\, h/2}$$

- Using $y_{i+1/2}$ calculate the slope at $x_{i+1/2}$.



- Use the slope at $x_{i+1/2}$ to calculate an estimate for y_{i+1}.

$$\mathbf{y_{i+1} = y_i + f(x_{i+1/2}, y_{i+1/2})\, h}$$

- Unlike Heun's Method, second step can NOT be applied more than once.
- If $f=f(x)$ only, than there is no need to perform the first step.
- Midpoint Method is 2nd order accurate. Its global error is $O(h^2)$.

Example

Solve the previous problem using the Midpoint Method with (a) h=0.5 , (b) h=0.25.

$f(x) = yx^2 - 1.2y$, $y_{i+1/2} = y_i + f(x_i, y_i)\, h/2$, $y_{i+1} = y_i + f(x_{i+1/2}, y_{i+1/2})\, h$

(a) $h = 0.5$

i=0 Predictor: $y_{0+1/2} = 1 + (1*0^2 - 1.2*1)\, 0.25 = 0.7$

Corrector: $y_1 = y(0.5) = 1 + (0.7*0.25^2 - 1.2*0.7)\, 0.5 =$ **0.601875**

i=1 Predictor: $y_{1+1/2} = 0.601875 + (0.601875*0.5^2 - 1.2*0.601875)\, 0.25 = 0.4589297$

Corrector: $y_2 = y(1) = 0.601875 + (0.4589297*0.75^2 - 1.2*0.4589297)\, 0.5 =$ **0.4555911**

i=2 Predictor: $y_{2+1/2} = 0.4555911 + (0.4555911*1^2 - 1.2*0.4555911)\, 0.25 = 0.4328116$

Corrector: $y_3 = y(1.5) = 0.4555911 + (0.4328116*1.25^2 - 1.2*0.4328116)\, 0.5 =$ **0.5340383**

i=3 Predictor: $y_{3+1/2} = 0.5340383 + (0.5340383*1.5^2 - 1.2*0.5350383)\, 0.25 = 0.6742233$

Corrector: $y_4 = y(2) = 0.5340383 + (0.6742233*1.75^2 - 1.2*0.6742233)\, 0.5 =$ **1.1619087**

Exercise Solve part (b) of the above example

Thanks

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