experiments are available for a number of metals and ϕ calculated from (7.43). The results are in good agreement with the Richardson–Dushman formula (7.49). For example Davisson and Germer found that for tungsten $W \simeq 13.5eV$, and since $\varepsilon_F \simeq 9eV$ for tungsten, we obtain $\phi \simeq 4.5eV$, in good agreement with the value of the work function as calculated from the slope of the experimental line.

7.4 MORE FERMI SYSTEMS

1. White dwarf stars

The properties of the electron gas proved to be of fundamental importance in explaining the stability of white dwarf stars, and this was historically the first application of Fermi statistics. In a star the source of energy is the hydrogen fusion reactions producing helium. When the hydrogen content of the star is exhausted, initial cooling and gravitational collapse may occur; in the final stages of stellar evolution there are three possibilities: formation of a white dwarf, or of a neutron star, or a total collapse to a black hole. Apparently the size of the star determines its final fate.

In a normal star release of nuclear energy by fusion keeps it inflated against gravity. White dwarf stars are faint because hydrogen having been burnt up the only energy source is from a gradual gravitational collapse. Consequently the density is high so that the core temperature is also high, of the order of 10^7 K, the temperature of the sun's core, accounting for their white colour. They are called dwarfs because though their masses are comparable to the sun's mass, their radii are only about one per cent of the solar radius. For the sun we have:

$$M_\odot = 2x10^{30} kg, \quad R_\odot = 7x10^8 m, \quad \rho_\odot \simeq 10^3 kg/m^3. \tag{7.50}$$

A typical white dwarf has a mass of the same order but its radius $R \sim 10^{-2} R_\odot$, hence its density,

$$\rho \simeq 10^6 \rho_\odot = 10^9 kg/m^3 \tag{7.51}$$

At such high densities the nuclei are denuded of their atomic electrons: the matter in white dwarfs consists of ionized He nuclei and free electrons. The electrons form an electron gas. The matter of the white dwarf consists of N electrons plus (1/2 N) He nuclei. Therefore its mass is

$$M \simeq N(m + 2m_p) \simeq Nm_p, \tag{7.52}$$

where m, m_p are the masses of an electron and a proton respectively. The concentration of the electrons

$$n = \frac{N}{V} = \frac{M/2m_p}{M/\rho} = \frac{\rho}{2m_p} \approx 10^{36}/m^3. \tag{7.53}$$

Such a high concentration of electrons means very large kinetic energies and hence very high pressure of the electron gas. Assuming complete degeneracy, the Fermi momentum (7.2) is therefore,

$$p_F = \left(\frac{3n}{8\pi}\right)^{1/3} h \approx 10^{-22}\, kg.m.\sec^{-1}. \tag{7.54}$$

The Fermi energy (7.3), putting $\gamma = 2$, is

$$\varepsilon_F = \left(\frac{3n}{8\pi}\right)^{2/3} \frac{h^2}{2m} \simeq 0.5\times10^{-13}\, J \simeq 3\times10^5\, eV, \tag{7.55}$$

hence, $T_F \sim 10^9\, K$. This is much higher than the core temperature $10^7\, K$, so the electron gas is effectively in the ground state. The question arises whether the electron energies are relativistic: the energy equivalence of the electron rest mass is

$$\varepsilon_0 = mc^2 \simeq (9\times10^{-31}\, kg)(3\times10^8\, ms^{-1})^2 \simeq 10^{-13}\, J \simeq 6\times10^5\, eV \tag{7.56}$$

of the same order as the Fermi energy. Hence relativistic effects will be present, but not dominant except at higher densities.

Apart from the electron gas there are the nuclei (mostly helium nuclei), but this system is not degenerate because the individual masses are much larger than an electron mass. The system remains a classical gas whose pressure is smaller than the electron gas by the ratio T/T_F, a negligible pressure.

The effect of the stars large mass and its exceedingly high density is to make the gravitational attraction between nuclei extremely strong. The high pressure of the degenerate electron gas (Sec. 7.3) supports the star against gravity and prevents further contraction. In the analysis of the stability or otherwise of the white dwarf, we need consider only the electron gas, the ionized nuclei have no effective role to play.

What is the requirement for stability of a white dwarf? The rigorous theory of white dwarfs was investigated by S. Chandrasekhar in the years 1931-35, but is rather difficult. However a rather crude simplification is sufficient to bring out the general conclusions. First we assume that the density of the star is uniform, and that the electron gas is distributed uniformly over the volume of the star. In other words, we ignore the spatial variation of density and other related parameters of the problem. The gravitational potential energy of the star, assumed to be spherical, is equal to $-\frac{3}{5}\frac{GM^2}{R}$, where G is the gravitational constant, and M is the mass of the star (7.52). To allow for a non-uniform density distribution, while taking the form of the expression to be valid, the numerical factor 3/5 is replaced by

α, another numerical constant. If the radius of the star is reduced by δR the gain in potential energy is equal to

$$\alpha \frac{GM^2}{R^2} \delta R .$$

(7.57)

If P is the pressure exerted by the electron gas in the star, the work done in collapsing the star is

$$PdV = P4\pi R^2 \delta R$$

(7.58)

The star is stable when the gain in gravitational potential energy is balanced by the work done against the pressure of the electron gas, that is,

$$P = \alpha \frac{G}{4\pi} \frac{M^2}{R^4}$$

(7.59)

The electron gas pressure depends on whether the electrons are in the relativistic regime or not. If the electrons are non-relativistic, the pressure is given by (7.5),

$$P = \frac{2}{3} n\varepsilon_F = \frac{2}{5} \left(\frac{3}{8\pi} \right)^{2/3} \left(\frac{N}{V} \right)^{5/3} \frac{h^2}{2m}\text{: and since } N = \frac{M}{2m_p}, \quad V = \frac{4}{3}\pi R^3 \text{, we write}$$

$$P = A \frac{M^{5/3}}{R^5}, \text{ where A is a constant given by } A = \frac{2}{5} \frac{h^2}{2m} \left(\frac{3}{8\pi} \right)^{7/3} m_p^{-5/3} \qquad (7.60)$$

The relation (7.59) becomes

$$A \frac{M^{5/3}}{R^5} = B \frac{M^2}{R^4}, \quad B = \alpha \frac{G}{4\pi} \text{, another constant, so that} \qquad (7.61)$$

$$M^{1/3} R = \frac{A}{B} = Cons\tan t \qquad (7.62)$$

This inverse relationship between mass and volume implies that a stable heavy white dwarf is ***smaller*** than a lighter one. A white dwarf is much smaller than a normal star of the same mass. A glance at the equation (7.61) shows that the left side grows as R^{-5} as R decreases, but the right side grows only as R^{-4}. Thus contraction enables the electron pressure to build up till it is able to support the star against gravity. If the relation $R \propto M^{-1/3}$ holds, the white dwarf is mechanically stable. This is true as long as the non-relativistic expression for the pressure (7.60) is valid, but the implication is that very heavy stars may never achieve stability. The heavier the star the smaller and the more dense it becomes: at high enough densities, the electrons approach relativistic speeds. The kinetic energy of an electron in the relativistic limit is then

$$\varepsilon = (m^2c^4 + p^2c^2)^{1/2} - mc^2 \approx pc, \tag{7.63}$$

or $\varepsilon \propto p$. The dispersion relation is the same as for photons, that is, r = 1 in (4.50) instead of $\varepsilon \propto p^2$ for non-relativistic electrons. Using the relations (7.1) through (7.4), with $\varepsilon = pc$, the total energy is given by,

$$E = \frac{8\pi V}{h^3} c \int_0^{p_F} p^3 dp = \frac{8\pi V}{h^3} c \frac{p_F^4}{4} = \frac{3}{4} N p_F c = \frac{3}{4} N \varepsilon_F \tag{7.64}$$

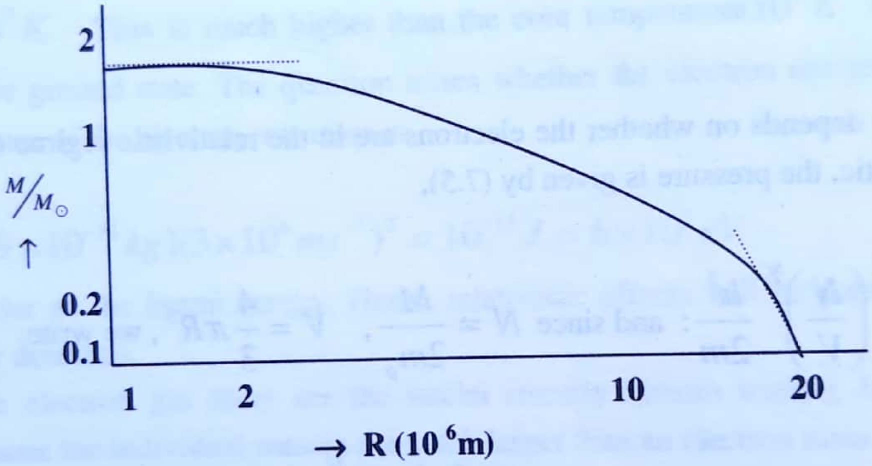

Fig. 7.12

The mass of a white dwarf as a function of its radius. The two dashed lines indicate the asymptotic limits of extreme relativistic and classical behaviour.

The pressure of the gas,

$$P = \frac{1}{3}\frac{E}{V} = \frac{1}{4}\frac{N}{V}\varepsilon_F = \frac{1}{4}\left(\frac{3}{8\pi}\right)^{1/3} hc \left(\frac{N}{V}\right)^{4/3} = D\left(\frac{M}{R^3}\right)^{4/3} \tag{7.65}$$

$$D = \frac{1}{4}\left(\frac{3}{8\pi}\right)^{1/3} hc, \text{ a constant.}$$

In equilibrium the pressure must balance the gravitational pressure (7.61), that is,

$$D\frac{M^{4/3}}{R^4} = B\frac{M^2}{R^4} \quad or \quad M^{2/3} = \frac{D}{B} \tag{7.66}$$

This is a strange result because the radius cancels out in (7.66): the mass saturates to a critical value independent of radius. Comparing (7.66) with (7.61), the stability criterion in the non-relativistic regime, one notices that on reaching the extreme relativistic regime, further contraction does not make the electron gas pressure increase faster than the gravitational attraction. If the mass does not satisfy (7.66) at once, the star cannot adjust its radius to achieve stability. The equation solves (putting in the values of constants B and D) for $M \sim 1.7M_{\odot}{}^{2}$. Taking into account the correct density distribution in the star, Chandrasekhar found the solution was $M = 1.44M_{\odot}$. This is called the Chandrasekhar limit, the largest possible mass of a stable white dwarf.